DIFFERENTIABLE MINIMIN SHAPE DISTANCE FOR INCORPORATING TOPOLOGICAL PRIORS IN BIOMEDICAL IMAGING

Yonggang Shi  William Clem Karl

Information Systems and Sciences Laboratory
Electrical and Computer Engineering Department
Boston University
Email: {yshi,wckarl}@bu.edu

ABSTRACT
In the application of curve evolution and level set methods to biomedical image analysis, the incorporation of geometric priors for isolated shapes has been proved useful. On the other hand, the inclusion of a priori topological information concerning the relationship of multiple shapes remains a challenge. In this paper, we propose a differentiable minimin shape distance (DMSD) that is indicative of the topological relation between shapes. A curve evolution equation based on its first variation is derived and this enables us to incorporate this prior into a curve evolution framework. We demonstrate the application of the DMSD by proposing an extension to the Chan-Vese image segmentation model to incorporate topological prior information for challenging image segmentation tasks.

1. INTRODUCTION
Curve evolution and level set methods are powerful techniques in image processing and analysis [1–5]. When applying these techniques to a specific field such as biomedical imaging, it is useful to incorporate various forms of prior knowledge, which has proved useful in many scenarios [6–9]. A common method of incorporating a shape prior in curve evolution methods is to compare the similarity or dissimilarity between shapes. While there are many distance measures that compare the geometric similarity between shapes [10], such as the symmetric difference between regions, the Hausdorff distances between point sets, or the difference between distance functions of curves, measuring the topological relation of multiple shapes and incorporating this measure into a curve evolution framework remains a challenging problem.

Toward this end, a topology preserving level set method is proposed in [11]. This is a numerical algorithm that can maintain the initial topology of the zero level set of a function as it evolves over time. But it does not have a mechanism to measure and incorporate more complicated topological relations between multiple shapes. For example, it can not handle the distance between two disconnected shapes and the variation of this distance with time. In this paper, we propose a novel differentiable minimin shape distance (DMSD) that is computable and indicative of topological relations between multiple shapes. Most importantly, it is differentiable and its gradients with respect the shapes can be computed. This enables us to incorporate it into the curve evolution framework, thereby including a topological prior in biomedical image analysis.

The rest of the paper is organized as follows. In Section 2, we propose our differentiable minimin shape distance (DMSD) and derive its gradients with respect to the shapes. In Section 3, we demonstrate the application of DMSD by proposing an extension of the Chan-Vese image segmentation model [4] to incorporate a priori topological relation between objects. Preliminary experimental results of segmenting a blood-pool image is also presented. Finally, conclusions are made in Section 4.

2. DIFFERENTIABLE MINIMIN SHAPE DISTANCE
Let $C_1$ and $C_2$ denote two curves in $\mathbb{R}^2$. Their minimin distance is defined as:

$$d_m(C_1, C_2) = \min_{x \in C_1} \left( \min_{y \in C_2} d(x, y) \right).$$

(1)

where $d(x, y)$ is the Euclidean distance between two points. Because we can exchange the order of the $\min$ operation in Eq.(1), this distance is symmetric with respect to $C_1$ and $C_2$. Using the distance functions of $C_1$ and $C_2$, denoted as $dc_1$ and $dc_2$ respectively, we can express the minimin distance as:

$$d_m(C_1, C_2) = \min_{x \in C_1} dc_2(x) = \min_{y \in C_2} dc_1(y).$$

(2)
As illustrated in Fig.1, \( d_m(C_1, C_2) \) is the shortest distance between points on \( C_1 \) and \( C_2 \). When the two curves intersect, it is zero. If the two curves are disconnected, we have \( d_m(C_1, C_2) > 0 \) and it tells us how far away the two curves are from each other. Thus it can be used to indicate the topological relation between \( C_1 \) and \( C_2 \). But there is one problem. Because this distance is defined through two \( \min \) operations, it is not differentiable with respect to both curves and this prevents us from incorporating it directly into a curve evolution framework.

Motivated by the differentiable approximations to Hausdorff distance in [10], we propose here a differentiable minimin shape distance (DMSD). If \( f \) is a positive and continuous function defined over the curve \( C_1 \), its \( L^p \) norm converges to its \( L^\infty \) norm as \( p \to \infty \), i.e.

\[
\lim_{p \to \infty} \left( \int_{C_1} f^p(x) \, dC_1(x) \right)^{1/p} = \sup_{x \in C_1} f(x). \tag{3}
\]

Based on this result and choosing \( f = \varphi \circ d_{C_2} \), where \( \varphi \) is a positive and strictly monotonically decreasing function, we have:

\[
d_m(C_1, C_2) = \lim_{p \to \infty} \varphi^{-1} \left[ \left( \int_{C_1} \varphi^p(d_{C_2}(x)) \, dC_1(x) \right)^{1/p} \right]. \tag{4}
\]

Choosing \( p > 0 \) and denoting \( \varphi_p(x) = \varphi^p(x) \), we define our DMSD as:

\[
\tilde{d}_m(C_1, C_2) = \varphi^{-1}_p \left( \int_{C_1} \varphi_p(d_{C_2}(x)) \, dC_1(x) \right). \tag{5}
\]

The first variation of \( \tilde{d}_m(C_1, C_2) \) with respect to \( C_1 \) is:

\[
\nabla_{C_1} \tilde{d}_m = \frac{\left( \nabla \varphi_p(d_{C_2}) \cdot \bar{N}_{C_1} - \varphi_p(d_{C_2}) \kappa_{C_1} \right) \bar{N}_{C_1}}{\varphi_p(d_{m})} \tag{6}
\]

where \( \bar{N}_{C_1} \) is the normal of \( C_1 \) and \( \kappa_{C_1} \) is its curvature.

In Fig.2, we illustrate the effect of evolving a curve \( C_1 \) according to Eq.(6). It shows that only those parts of \( C_1 \) that are the closest to \( C_2 \) move away from \( C_2 \) while other parts remain static. This demonstrates that \( \tilde{d}_m(C_1, C_2) \) is a good approximation of \( d_m(C_1, C_2) \) and we can use the corresponding curve evolution equation to control the topological relation between shapes.

Fig. 2. The curve evolution process of \( C_1 \) (the left curve) to maximize its DMSD to \( C_2 \) (the right curve): (a)-(f).

3. APPLICATIONS IN BIOMEDICAL IMAGING

In this section, we demonstrate the application of the DMSD in biomedical imaging. We propose an extension of the Chan-Vese image segmentation model [4] to incorporate topological prior information into the energy function.

3.1. The Chan-Vese Model

The Chan-Vese active contour model [4] segments an image into two regions by the minimization of an energy function. Let \( f(x, y) \) denote the noisy image and assume that it is composed of two regions: the object region \( \Omega \) and the background region \( \Omega_b \). The intensities of these two regions are \( \beta_{in} \) and \( \beta_{out} \) and the object boundary is denoted as \( C \). The energy function is then defined as follows:

\[
E(\beta_{in}, \beta_{out}, C) = \int_{\Omega} |f(x, y) - \beta_{in}|^2 \, dx \, dy + \lambda \int_{\Omega_b} |f(x, y) - \beta_{out}|^2 \, dx \, dy + \mu ||C|| + \nu \int_{\Omega} dx \, dy \tag{7}
\]

where \( ||C|| \) is the length of the boundary, \( \int_{\Omega} dx \, dy \) is the area of the object region, and \( \lambda, \mu, \nu \) are non-negative regularization parameters. By minimizing the energy, we find the best fit of the two-region model to the original image \( f(x, y) \) and segmentation is achieved.
3.2. The Incorporation of A Topological Prior

Here we propose an extension of the Chan-Vese model to incorporate topological prior knowledge among objects using the DMSD proposed in Section 2.

Without loss of generality, we present our model in the case that the object region is further composed of two disconnected regions: $\Omega_1$ and $\Omega_2$. The extension of our model to include multiple disconnected object regions is straightforward. The boundaries of these two regions are denoted as $C_1$ and $C_2$, and our new energy function is defined as follows:

$$
E(\beta_{in}, \beta_{out}, C_1, C_2) = 
\int_{\Omega_1} |f(x, y) - \beta_{in}|^2 dx dy + \int_{\Omega_2} |f(x, y) - \beta_{in}|^2 dx dy 
+ \lambda \int_{\partial \Omega_2} |f(x, y) - \beta_{out}|^2 dx dy + \mu ||C|| + \nu \int_{\Omega} dxdy 
+ \frac{\xi}{d_m(C_1, C_2)}
$$

(8)

where $d_m(C_1, C_2)$ is the DMSD between the two curves $C_1$ and $C_2$, $l \geq 1$ is an exponent, and $\xi$ is a non-negative regularization parameter. Compared with the Chan-Vese energy in Eq.(7), the novelty of our energy function is that we have added a term that is inversely proportional to $d_m(C_1, C_2)$. This term will approach infinity if the DMSD between $C_1$ and $C_2$ approaches zero, thus they can not get too close to each other and their topological relation is maintained.

To minimize our energy function, we use a coordinate descent algorithm. In the first step of the algorithm, we estimate $\beta_{in}$ and $\beta_{out}$ to be the mean of the image $f$ in the two object regions and the background region respectively. Then we evolve $C_1$ and $C_2$ in the gradient descent direction with level set methods [2] in the second step. The curve evolution equation is given as follows:

$$
dC_1 \over dt = \left[ \lambda (f - \beta_{out})^2 - (f - \beta_{in})^2 + \mu \kappa_{c_1} - \nu + \frac{\xi l \left( \nabla \varphi_p(d_c) \cdot \vec{N}_{c_1} - \varphi_p(d_c) \kappa_{c_1} \right)}{d_{m+1}(C_1, C_2) \varphi'_p(d_m)} \right] \vec{N}_{c_1}
$$

(9)

$$
dC_2 \over dt = \left[ \lambda (f - \beta_{out})^2 - (f - \beta_{in})^2 + \mu \kappa_{c_2} - \nu + \frac{\xi l \left( \nabla \varphi_p(d_c) \cdot \vec{N}_{c_2} - \varphi_p(d_c) \kappa_{c_2} \right)}{d_{m+1}(C_1, C_2) \varphi'_p(d_m)} \right] \vec{N}_{c_2}
$$

(10)

where $\kappa_{c_1}$ and $\kappa_{c_2}$ are the curvatures of $C_1$ and $C_2$, $\vec{N}_{c_1}$ and $\vec{N}_{c_2}$ are their normals, and $d_{c_1}$ and $d_{c_2}$ are their distance functions. When the two curves get close, the last term in Eq.(9) and (10) will prevents them from touching each other, thus each curve will stay away from the other curve during the energy minimization process if we choose the initial curves to be disconnected.

3.3. Experimental Results

We present some preliminary experimental results on the application of our model to the segmentation of blood-pool images in nuclear medicine [12]. The challenge here is that the boundary between the blood pools of the left and right ventricle are unclear because the images are usually blurred and with low resolution. Our goal is to investigate the advantages of incorporating the topological prior knowledge that the blood pools to be segmented are two disconnected objects.

The original image to be segmented is shown in Fig.3. It is simulated from the MCAT phantom [13], a widely used mathematical phantom in nuclear medicine study. We applied both the Chan-Vese model and our method to segment this image. Both methods start with the same initial curves as shown in Fig.4. The final segmentation result of the Chan-Vese model is shown in Fig.5. We can see that this method is able to segment out the object region from the background region, but it fails to find out the boundaries between the two objects because they are smeared into each other and almost invisible. The segmentation result with our method is shown in Fig.6, and it demonstrates that our method successfully maintains the original topology of the initial curves and clearly finds the boundaries between the left and right blood-pool.

4. CONCLUSIONS

In this paper, we proposed a differentiable minimin shape distance(DMSD) that is indicative of the topological relations between multiple shapes and curve evolution equations for this distance is derived. Using the DMSD, we proposed an extension to the Chan-Vese image segmentation model to incorporate the topological prior information in biomedical images. Preliminary results demonstrate that our method can segment out adjacent objects successfully with the help of a topological prior.

5. REFERENCES


Fig. 3. The original blood-pool image simulated from the MCAT phantom [13].

Fig. 4. The initial curves for both methods.

Fig. 5. The segmentation result of the Chan-Vese model.

Fig. 6. The segmentation result of our method with topological prior.


