High Rate Codes with Bounded PMEPR for BPSK and Other Symmetric Constellations

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Abstract

In this letter, we consider the problem of constructing high rate codes with low peak to mean envelope power ratio (PMEPR) for multicarrier signals. Assuming coefficients of the multicarrier signal are chosen from a symmetric q-ary constellation, we construct codes with rate $1-\frac{1}{r}\log_q 2$ and PMEPR of less than $cr\log n$ for any r and n, where n is the number of subcarriers and c is a constant independent of n and r. The construction is based on dividing n subcarriers into n/r groups of r subcarriers and choosing a sign for each group to minimize the PMEPR. The signs are chosen using a variation of the algorithm proposed in [1], [2]. For large n, we can in fact construct a code with a rate of $1-O(1/\log n)$ and PMEPR of less than $c\log^2 n$. For BPSK modulated signals, this partially solves the problem posed by Litsyn [3] and implies a construction of $2^{n/2}$ codewords with PMEPR less than $2c\log n$.

I. Introduction

Multicarrier signals are proposed in many high speed wireless and wireline standards such as WLAN 802.11, xDSL, and DVB. In a typical multicarrier signal, the transmitted signal consists of a large number of subcarriers (e.g. 64 for the WLAN standard) that leads to a signal with high peak to mean envelope power ratio (PMEPR). This large PMEPR requires a highly linear power amplifier in the transmitter front end that consequently hampers its power efficiency.

Over the years, a whole host of methods have been developed for PMEPR reduction such as coding, deliberate clipping, selective mapping (SLM), reserved carriers, and tone injection [4], [5], [6], [7], [8], [9], [10]. Of course PMEPR reduction comes at a price in terms of coding rate, average power, signal distortion, and bandwidth. Methods like coding usually give a worst case guarantee on the PMEPR. On the other hand, there are other methods such as SLM and partial transmit sequence (PTS) that improve the probability distribution of the PMEPR, i.e. reduce the probability of encountering a large

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PMEPR. In this paper, we propose a coding scheme that reduces the PMEPR at the expense of a rate hit.

Recently, in [1], an algorithm has been proposed to choose the sign of each subcarrier in order to reduce the PMEPR. In other words, for a symmetric q-ary constellation, a code is constructed with rate $1 - \log_q 2$ and PMEPR of less than $c \log n$ for any n, where c is a constant independent of n. The algorithm works only for constellations larger than BPSK, i.e. $q > 2$. For BPSK constellations, recently in [10], the authors characterized the codewords with PMEPR higher than $\frac{3}{4}n$. This motivates the question of whether we can construct exponentially many codewords from BPSK with PMEPR bounded by $\log n$ [3].

In this letter, we partially solve the aforementioned problem. We propose an algorithm that only uses the signs of $\frac{n}{r}$ subcarriers and guarantees that the PMEPR of the resulting codeword is less than $cr \log n$ for any n and any r [1]. For a BPSK constellation, this implies a code with rate $1 - \frac{1}{r}$ and PMEPR less than $cr \log n$. In general for a symmetric q-ary constellation, we construct a code with rate $1 - \frac{1}{r} \log_q 2$ and a PMEPR of less than $cr \log n$. It is also worth noting that all the complexity in the encoding is to find the optimum signs, which can be done in polynomial time, and the decoding is also simple as the transmitter does not send any information over the signs of $\frac{n}{r}$ subcarriers. An interesting consequence of this result is the construction of codes with rate $1 - O(1/\log n)$ (almost full rate) and PMEPR of less than $c \log^2 n$. Simulation results show a significant PMEPR reduction with a little rate loss.

The letter is organized as follows: Section 2 introduces our notation and the statement of the problem. In Section 3, the main result of the paper is presented and high rate codes with bounded PMEPR is constructed. Section 4 discusses extensions of our work and open problems. Simulation results are presented in Section 5 and Section 6 concludes the paper.

II. Definitions and Problem Statement

We may write the normalized multicarrier signal $s_C(\theta)$ that consists of n subcarriers as,

$$s_C(\theta) = \sum_{i=1}^{n} c_i e^{j\theta i}, \tag{1}$$

where $C = (c_1, \ldots, c_n)$ is the modulating vector (or the codeword), c_i's are chosen from a constellation such as BPSK or QAM, and θ denotes time. Clearly, if the c_i's are chosen from an MPSK constellation

[1]Throughout the paper, c is a constant independent of n and r.

and they add up coherently, $s_C(\theta)$ will have a large peak of amplitude n.

In order to evaluate the variation of the amplitude of a multicarrier signal, we may define the PMEPR of C as

$$\mathrm{PMEPR}(C) = \max_{0\leq\theta\leq 2\pi} \frac{|s_C(\theta)|^2}{E\{\sum_{i=1}^n |c_i|^2\}}, \tag{2}$$

where the denominator is the average power of $s_C(\theta)$. Thus, if c_i's are chosen independently from a constellation with average power of P_{av}, then $E\{\sum_{i=1}^n |c_i|^2\} = nP_{av}$.

In this paper, we consider PMEPR reduction by adjusting the sign of *some* of the subcarriers. This method is a more general version of the scheme that has been recently proposed in [1] and it is shown that by adjusting the signs of *all* subcarriers, we can achieve constant PMEPR with a little rate hit for large n [11]. In [2] and [1], an algorithm is also proposed to design the signs such that the resulting PMEPR is guaranteed to be less than $c\log n$ for any n.

Therefore, by using the sign of each subcarrier to reduce the PMEPR, we can construct a code with $(q/2)^n$ codewords (out of q^n) that has a PMEPR of less than $c\log n$ [2]. To quantify this reduction in the number of codewords, we may define the rate of a code $\mathcal{C}$ as

$$R = \frac{1}{n}\log_q |\mathcal{C}|, \tag{3}$$

where c_i's are chosen from a q-ary constellation and $|\mathcal{C}|$ denotes the cardinality of the set $\mathcal{C}$. Clearly based on our definition of rate, full rate codes referes to $R = 1$ for any q.

It has been shown that almost all the q^n codewords have a PMEPR of less than $\log n + O(\log\log n)$ for various kinds of constellations including symmetric ones[2] when n is large [12], [13], [14]. However, there is no explicit construction of an almost rate-one code that approaches that limit for PMEPR. For BPSK, there is even no construction of exponentially many codewords with PMEPR less than $c\log n$ [3].

In this paper, we address both of the aforementioned problems. Namely, we construct codes that have almost full rate (i.e. $R = 1 - O(\frac{1}{\log n})$) and a PMEPR of less than $c\log^2 n$ for large n. Furthermore, for BPSK constellations, we propose codes with rate $R = 1 - \frac{1}{r}$ and PMEPR less than $cr\log n$.

III. HIGH RATE CODES WITH BOUNDED PMEPR

Unlike [2], where the signs of *all* subcarriers are used to reduce the PMEPR, we consider the problem of using a fraction of all the signs to reduce the PMEPR. This is in fact motivated by the fact

[2]by symmetric constellation, here we mean that if a point A belongs to the constellation, $-A$ will also be in the constellation.

that for BPSK there is no construction for a high rate code with PMEPR of less than $O(\log n)$ (see [10] and [3]) and the scheme of [2] does not work for BPSK as it uses the signs of all subcarriers to reduce the PMEPR.

In order to construct BPSK codewords with bounded PMEPR, for any integer r, we use the sign of $\frac{n}{r}$ equally spaced subcarriers indexed from 1 to n to minimize the PMEPR [3]. Therefore, for any given codeword C, we would like to solve the following non-convex problem:

$$\min_{\epsilon_1,\epsilon_2,\ldots} \max_{0\le\theta\le 2\pi} \left| \sum_{i=0}^{\frac{n}{r}-1} \epsilon_{i+1} \left(\sum_{s=1}^{r} c_{ir+s} e^{j\theta(ir+s)} \right) \right| \tag{4}$$

where $\epsilon_i \in \{+1,-1\}$. It is worth mentioning that setting $r = 1$ reduces the problem to the one addressed in [2].

Following [2], instead of maximizing over $0 \le \theta \le 2\pi$, we can minimize the maximum over kn samples of θ [15], [16]. Therefore the min-max problem of (4) can be written as,

$$\min_{\epsilon_1,\epsilon_2,\ldots} \max_{1\le p\le 2kn} \left| \sum_{i=0}^{\frac{n}{r}-1} \epsilon_{i+1} a_{pi} \right| \tag{5}$$

where

$$a_{pi} = \begin{cases} \mathrm{Re}\{\sum_{s=1}^{r} c_{ir+s} e^{j\theta(ir+s)}\} & 1 \le p \le kn, \\ \mathrm{Im}\{\sum_{s=1}^{r} c_{ir+s} e^{j\theta(ir+s)}\} & kn+1 \le p \le 2kn, \end{cases} \tag{6}$$

and $\theta_p = \frac{2\pi p}{kn}$.

Fortunately, the machinery used in [2] can be generalized to this case and the following algorithm can be deduced by using a derandomization method as what we did in [2].

Algorithm 1. *For any $C = (c_1, \ldots, c_n)$, let k be an integer greater than 1 and $|c_i| \le \sqrt{E_{max}}$. Then $\epsilon_1 = 1$, and ϵ_s's are recursively determined as*

$$\epsilon_j = -\mathrm{sign} \left\{ \sum_{p=1}^{2kn} \sinh \left\{ \alpha^* \sum_{r=1}^{s-1} \epsilon_r a_{pr} \right\} \sinh(\alpha^* a_{ps}) \prod_{r=s+1}^{n/r} \cosh\{\alpha^* a_{pr}\} \right\}.$$

for $s = 2, \ldots, \frac{n}{r}$, where $\alpha^* = \sqrt{\frac{2\log 4kn}{nrE_{max}}}$.

The next Theorem gives a guarantee on the PMEPR of the resulting codeword by using Algorithm 1.

[3]For simplicity, we always assume r divides n. This condition is not necessary and it is just for simplifying the notations.

Theorem 1: Let C be a given codeword where $c_i \leq \sqrt{E_{max}}$ and $E_{av} = E\{|c_i|^2\}$. Also let $C_\epsilon = (\epsilon_1 c_1, \ldots, \epsilon_1 c_{r-1}, \epsilon_2 c_r, \ldots)$ where ϵ_i's are determined according to Algorithm 1. Then the PMEPR of C_ϵ is less than $\frac{4rE_{max}}{\cos^2(\pi/2k)E_{av}} \log 4kn$ for any n and r where k is a positive integer such that kn is an integer.

Proof: The proof is along the same line as the proof of Theorem 3 of [2]. The only difference here is that we are minimizing the maximum of $2kn$ linear forms over n/r signs as opposed to n in [2]. ■

Remark 1: It is worth mentioning that our scheme is similar to the PTS method in that we search for the optimum sign for each group to minimize the PMEPR. The difference however is that we do not require side information in the receiver as the signs that used for PMEPR reduction do not carry any information. Moreover, we propose a simple deterministic algorithm that provides a guarantee on the PMEPR without performing any Fourier transformation [8].

Remark 2: Since k is a constant, Theorem 1 implies that the resulting codeword has a PMEPR of less than $c \log n$ where c is a constant independent of n and r and that c can be determined by optimizing over k.

We can now construct a code set $\mathcal{C}$ such that the PMEPR of all its codewords is less than $cr \log n$ when the c_i's are chosen from a symmetric q-ary constellation. This can be done by reserving the sign of only n/r subcarriers (indexed $i = 1, r+1, 2r+1, \ldots, n-r+1$) to minimize the PMEPR over those signs. Given all the c_i's, Algorithm 1 can be used to determine the signs in polynomial time. Therefore, we end up having $(q/2)^{n/r} q^{n-n/r}$ codewords with the PMEPR of less than $cr \log n$ for any n and r. That leads to the following Corollary:

Corollary 1: If c_i's are chosen from a q-ary constellation, the code $\mathcal{C}$ constructed using Algorithm 1 has a rate of $1 - \frac{1}{r} \log_q 2$ and its PMEPR is less than $cr \log n$ for any n and r, where c is a constant independent of n and r.

It is worth noting that the decoding of $\mathcal{C}$ is quite simple as the decoder can infer the signs (ϵ_i's) from the corresponding subcarriers since they do not convey any information over their signs.

Remark 3: The extension of our algorithm to the case where ϵ_i's can be chosen from $\{\pm 1 \pm j\}$ is straightforward. In this case ϵ_i's in (4) should be replaced by $\epsilon_i + j\epsilon'_i$ where ϵ_i and ϵ'_i are chosen from $\{+1, -1\}$. Therefore using the same argument as in (5), the problem can be again written in a similar form as in (5) and can be solved using Algorithm 1.

In the next two subsections, we further look into two special cases, namely, we look into codes for

BPSK constellations and also the PMEPR-rate region of the codes when n is large.

A. Example: BPSK Constellations

The famous result of Halasz [12] states that almost all BPSK codewords have a PMEPR of less than $\log n + O(\log\log n)$ for large n. The design of such a code has been recently addressed in [10] where codewords with PMEPR of less than $\frac{3}{4}n$ have been characterized for any n.

Corollary 1 in fact constructs $2^{n(1-1/r)}$ codewords with PMEPR of less than $cr\log n$ for any n. For fixed r and large n, this implies a construction of exponentially many codewords (in fact $2^{n(1-1/r)}$) such that their PMEPR is $O(\log n)$.

B. Large Number of Subcarriers

In [14], the result of Halasz is extended to many other constellations including symmetric QAM, PSK constellations and spherical codes. Therefore, the existence of codes with rate approaching one and the PMEPR of less than $\log n + O(\log\log n)$ has been established, although, there is no construction close to this result [3].

Since Algorithm 1 and Theorem 1 work for any n and r, we may choose r to be $\log n$. We can therefore use Corollary 1 to prove that we can construct a code with rate $1 - O(\frac{1}{\log n})$ and a PMEPR of less than $c\log^2 n$. We can make a more precise argument in the following Corollary.

Corollary 2: For large n, using the construction as in Corollary 1, the code $\mathcal{C}$ has a rate $1 - O(\frac{1}{f(n)})$ and a PMEPR of less than $cf(n)\log n$ where $1 \leq f(n) \leq n$ such that $\lim_{n\to\infty} f(n) = \infty$.

Therefore in an attempt to construct almost rate-one codes with PMEPR of $\log n$, we have been able to construct codes with almost rate-one (in fact, $R = 1 - O(\frac{1}{f(n)})$) and PMEPR of less than $cf(n) \times \log n$ where c is a constant independent of n.

IV. Extensions and Open Problems

In [17], [18], constellation shaping is proposed to reduce the PMEPR. In [17], it is shown that by using amplitude adjustment on top of adjusting all the signs, i.e. $r = n$, PMEPR can be further reduced at the expense of a little average power penalty. We can therefore use the same approach and combine our coding scheme with amplitude adjustment to further reduce the PMEPR at the expense of a lower rate hit and a little average power increase. It would be interesting to see what are the best rate and average power increase that lead to the maximum PMEPR reduction at the minimum cost.

The problem of constructing almost rate-one codes with PMEPR of $\log n + O(\log\log n)$ still remains open. Even though the result of this letter implies a construction of a code with rate almost one (i.e. $R = 1 - O(\frac{1}{\log n})$) and PMEPR of less than $c\log^2 n$, we are still off by a factor of $\log n$.

Another interesting problem is to investigate the performance of our algorithm (or any other algorithm) for designing the signs. Therefore, it would be useful to provide a lower bound on the value of the optimization problem posed in (5) as a bench mark. Of course one simple lower bound can be obtained by using Parseval's theorem and bounding the maximum of the signal by its average power. However, obtaining tighter bounds remains as an open problem.

Furthermore, as shown in [2], there exist non-vanishing to zero rate codes with even constant PMEPR for large n. Achieving codes with better PMEPR calls for a sign algorithm that has a better performance (or bound) for PMEPR. This also remains as a very interesting open problem.

V. Simulation Results

As we discussed in the previous sections, there is a trade-off between the PMEPR reduction and the rate of a code. In this section we carry out simulations to explore this trade off for $n = 128$ and for BPSK, QPSK, and 16QAM constellations. The algorithm for designing the signs is applicable to any symmetric constellation.

Fig. 1 shows the CCDF (complementary cumulative distribution function) of the PMEPR when c_i's are chosen from a BPSK constellation and for different values of r which correspond to rate $1/2$ and $3/4$ codes. As we use fewer signs (or increase the rate of code), the CCDF shifts to the right.

Fig. 2 and 3 shows the CCDF of PMEPR for QPSK and 16QAM constellations. For QPSK curves refer to different coding rates including $1/2$, $3/4$, and $7/8$, and for 16QAM, rates are $3/4$, $7/8$, and $15/16$. It is also worth noting that although the rate of the code highly depends on the constellation, for a fixed value of r, the CCDF does not really change as we change the constellation from BPSK to 16QAM. Therefore, PMEPR reduction comes with a lower penalty in terms of rate for higher order constellations.

VI. Conclusion

This paper addresses the design of high rate codes with bounded PMEPR. By using the signs of n/r subcarriers and when the c_i's are chosen from a q-ary symmetric constellation, we constructed codes with rate $1 - \frac{1}{r}\log_q 2$ and PMEPR of less than $cr\log n$ for any n and r. For BPSK constellations this

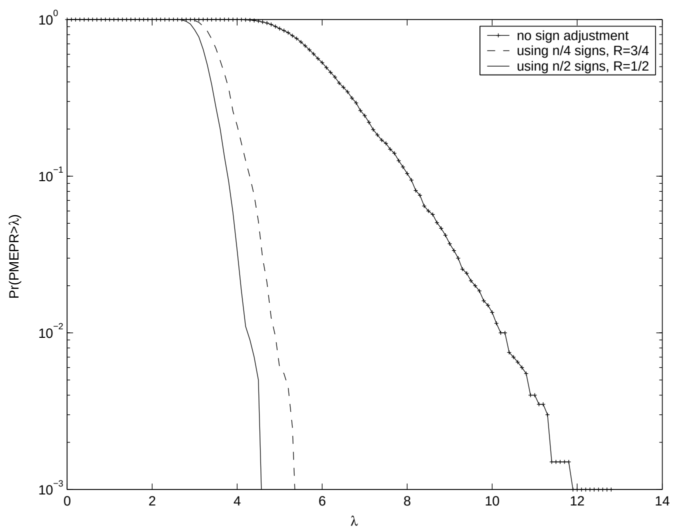

Fig. 1. CCDF of the PMEPR for BPSK by optimizing over n/r signs for $r = 2, 4$ when $n = 128$. This corresponds to codes with rate $1/2$ and $3/4$.

implies codes with exponentially many codewords (in fact $2^{n(1-1/r)}$) that have a PMEPR of less than $cr \log n$ where c is a constant independent of n and r. For large n, our result implies a construction for a code with almost full rate, i.e. $1 - O(\frac{1}{\log n})$, and the PMEPR of $c \log^2 n$. It is also worth noting that the encoding can be done in polynomial time and the decoding is quite simple.

References

[1] M. Sharif and B. Hassibi, "A deterministic algorithm that achieves PMEPR of $c \log n$ for multicarrier signals," *Proc. IEEE ICASSP*, pp. 540–543, 2003.

[2] M. Sharif and B. Hassibi, "Existence of codes with constant PMEPR and related designs," *IEEE Trans. Signal Processing*, vol. 52, no. 10, Oct. 2004.

[3] S. Litsyn, "On the PAPR problem," in *DIMACS Workshop on Algebraic Coding Theory and Information Theory*, Dec. 15–18 2003.

[4] K .G. Paterson and V. Tarokh, "On the existence and construction of good codes with low peak to average power ratios," *IEEE Trans. Inform.*, vol. 46, no. 6, pp. 1974–1986, Sep. 2000.

[5] K .G. Paterson, "Generalized Reed-Muller codes and power control in OFDM modulation," *IEEE Trans. Inform.*, vol. 46, no. 1, pp. 104–120, Jan. 2000.

[6] H. Ochiai and H. Imai, "Performance of the deliberate clipping with adaptive symbol selection for strictly band-limited OFDM systems," *IEEE Jour. Selec. Areas Commu.*, vol. 18, no. 11, pp. 2270–2277, Nov. 2000.

[7] X. Li and L. J. Cimini, "Effects of clipping and filtering on the performance of OFDM," in *Proc. IEEE Veh. Tech. Conf.*, May 1997, pp. 1634–1638.

[8] S. H. Muller and J. B. Huber, "A comparison of peak power reduction schemes for OFDM," in *Proc. IEEE Glob. Comm. Conf.*, 1997, pp. 1–5.

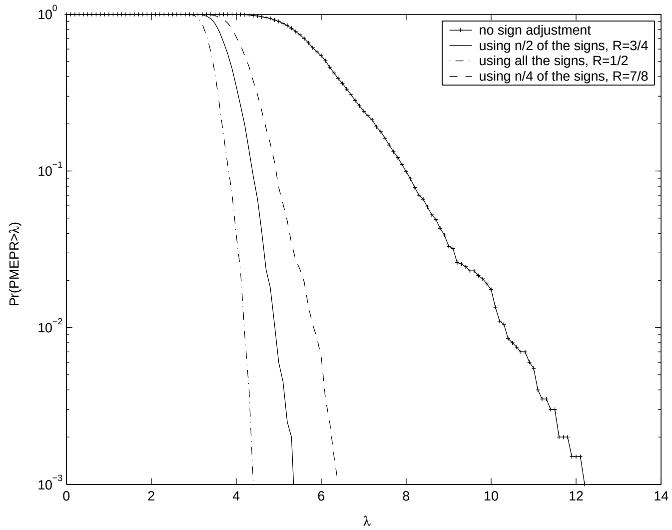

Fig. 2. CCDF of the PMEPR for QPSK by optimizing over n/r signs for $r = 1, 2, 4$ when $n = 128$. This corresponds to codes with rate $1/2$, $3/4$, and $7/8$.

[9] J. Tellado, *Multicarrier modulation with low PAR: applications to DSL and Wireless*, Kluwer Academic Publishers, 2000.

[10] G. Freiman, S. Litsyn, and A. Yudin, "A method to suppress high peaks in BPSK modulated OFDM signal," *IEEE Trans. Communications*, vol. 52, no. 9, Sep. 2004.

[11] M. Sharif and B. Hassibi, "On the existence of codes with constant peak to mean envelope power ratio," *Proc. IEEE Int. Symp. Info. Theo.*, 2003.

[12] G. Halasz, "On the result of Salem and Zygmund concerning random polynomials," *Studia Scien. Math. Hung.*, pp. 369–377, 1973.

[13] G. Wunder and H. Boche, "Upper bounds on the statistical distribution of the crest factor in OFDM transmission," *IEEE Trans. Info. Theory*, vol. 49, no. 2, pp. 488–494, Feb. 2003.

[14] M. Sharif and B. Hassibi, "On multicarrier signals where the PMEPR of a random codeword is asymptotically $\log n$," *IEEE Trans. Inform.*, vol. 50, no. 5, May.

[15] G. Wunder and H. Boche, "Peak value estimation of bandlimited signals from their samples, noise enhancement, and a local characterization in the neighborhood of an extermum," *IEEE Trans. Signal Processing*, vol. 51, no. 3, pp. 771–780, March 2003.

[16] H. Ehlich and K. Zeller, "Schwankung von polynomen zwischen gitterpunkten," *Math. Zeitschr.*, vol. 86, pp. 41–44, 1964.

[17] M. Sharif, C. Florens, M. Fazel, and B. Hassibi, "Peak to average power reduction using amplitude and sign adjustment," in *in Proc. IEEE Int. Conf. Commun.*, 2004.

[18] D. Jones, "Peak power reduction in OFDM and DMT via active channel modification," in *Proc. of 32nd Asilomar Conf.*, Nov. 1999, pp. 1076–1079.

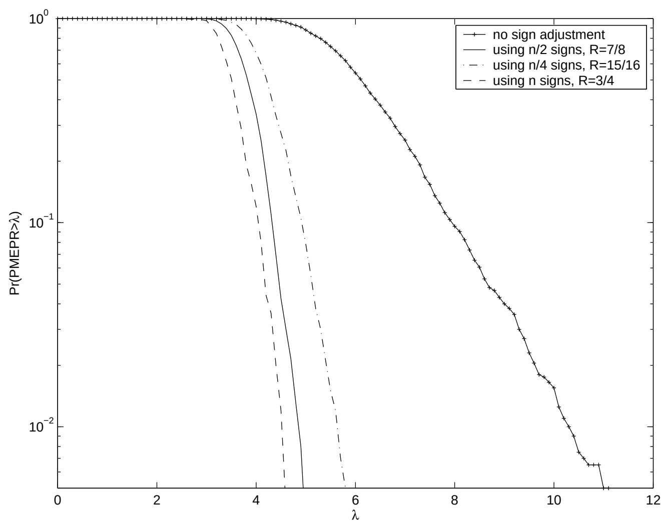

Fig. 3. CCDF of the PMEPR for 16QAM by optimizing over n/r signs for $r = 1, 2, 4$ when $n = 128$. This corresponds to codes with rate $3/4$, $7/8$, and $15/16$.