Integer-Forcing: An Algebraic Approach to Interference Management

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Boston University

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for help with the plots.

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Interference Management Session
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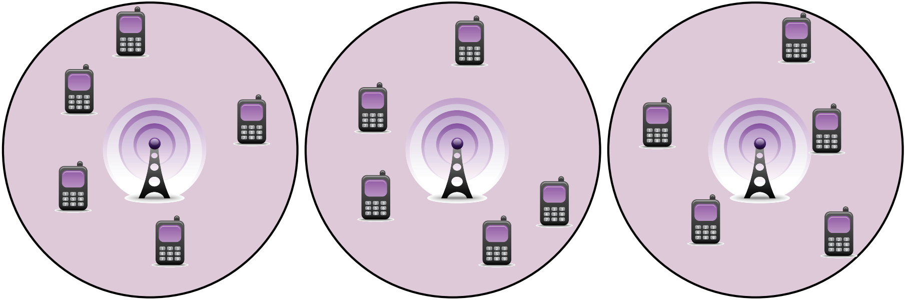
Motivation



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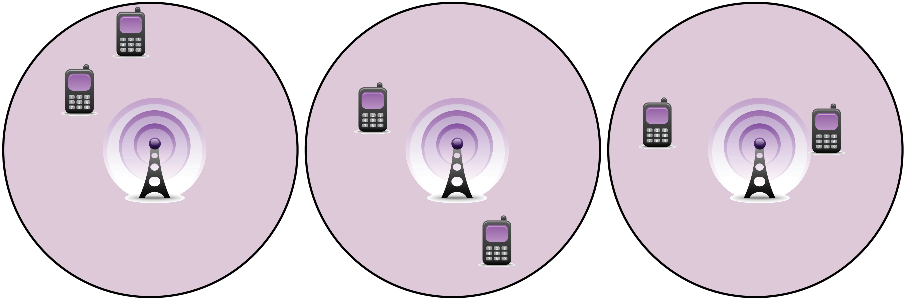

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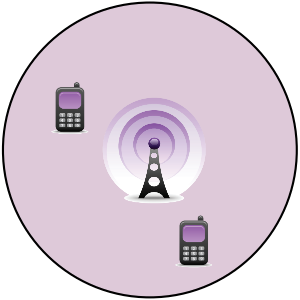
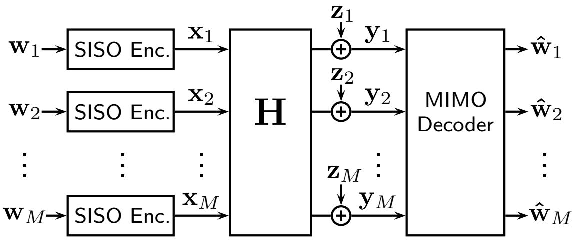


MIMO Uplink Channel



Usual Assumptions:

- Each antenna carries an independent data stream $\mathbf{x}_\ell \in \mathbb{C}^n$ of rate R (e.g., V-BLAST setting, cellular uplink). $\mathbf{X} = [\mathbf{x}_1 \ \cdots \ \mathbf{x}_M]^\mathsf{T}$.
- Usual power constraint: $\|\mathbf{x}_\ell\|^2 \leq n\mathsf{SNR}$.
- Channel model: $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{Z}$
- $\mathbf{Z}$ is elementwise i.i.d. $\mathcal{CN}(0,1)$.
- CSIR: Only the receiver knows channel realization $\mathbf{H} \in \mathbb{C}^{M \times M}$.

Outage Rates and Probabilities

- Throughout the talk, we will assume that $\mathbf{H}$ is elementwise i.i.d. Rayleigh, remains fixed throughout the block, and is only known at the receiver.

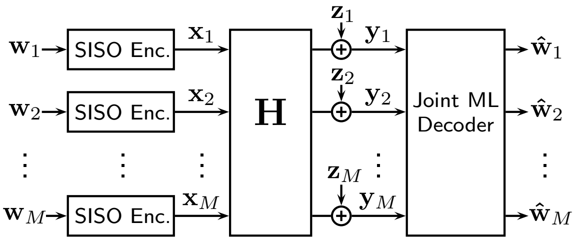
- Say that we have a scheme that achieves rate $R_{\text{scheme}}(\mathbf{H})$ under channel realization $\mathbf{H}$. For a target rate R, the outage probability is

$$p_{\text{out}}(R) = \mathbb{P}\big(R_{\text{scheme}}(\mathbf{H}) < R\big)$$

  and the outage rate is

$$R_{\text{out}}(\rho) = \sup\big\{R : p_{\text{out}}(R) \leq \rho\big\}.$$

MIMO Uplink Channel: Joint ML Decoding

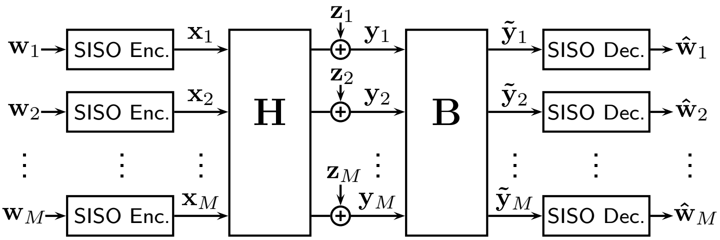


Joint Maximum Likelihood Decoding:

$$R_{\mathsf{joint}}(\mathbf{H}) = \min_{\mathcal{S} \subseteq \{1,\dots,M\}} \frac{1}{|\mathcal{S}|} \log\det\left(\mathbf{I} + \mathsf{SNR}\ \mathbf{H}_{\mathcal{S}}\mathbf{H}_{\mathcal{S}}^*\right)$$

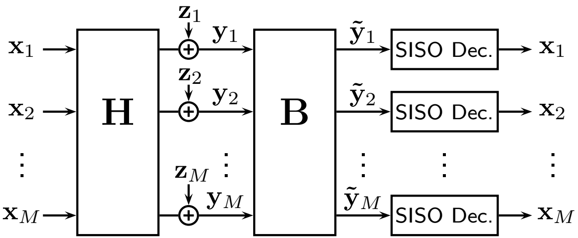
- Corresponds to the (symmetric) outage capacity.
- Naive implementation has prohibitively high complexity.
- Of course, there are many clever ways to reduce the complexity!

MIMO Uplink Channel: Zero-Forcing and Linear MMSE



Zero-Forcing and Linear MMSE Receivers:

- Project the received signal, $\tilde{\mathbf{Y}} = \mathbf{B}\mathbf{Y}$ to eliminate interference between data streams.
- After projection, single-user decoders attempt to recover the individual data streams.
- Optimal $\mathbf{B}$ is the MMSE projection.



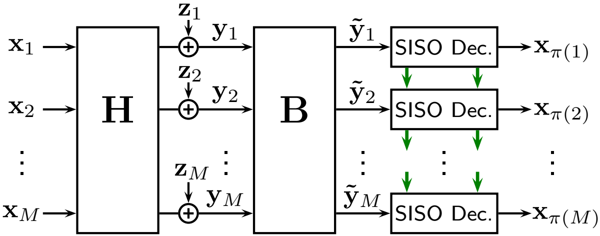
Zero-Forcing and Linear MMSE Receivers:

- The m^{th} SISO decoder tries to recover $\mathbf{x}_m$ from $\mathbf{b}_m^\mathsf{T}\mathbf{Y}$:

$$\mathsf{SINR}_{\mathsf{LMMSE},m}(\mathbf{H}) = \max_{\mathbf{b}_m} \frac{\mathsf{SNR}\ \|\mathbf{b}_m^\mathsf{T}\mathbf{h}_m\|^2}{1 + \mathsf{SNR}\ \sum_{\ell \neq m} \|\mathbf{b}_m^\mathsf{T}\mathbf{h}_\ell\|^2}$$

- Rate per user:

$$R_{\mathsf{LMMSE}}(\mathbf{H}) = \min_{m=1,\ldots,M} \log\left(1 + \mathsf{SINR}_{\mathsf{LMMSE},m}(\mathbf{H})\right)$$



Successive Interference Cancellation Receivers:

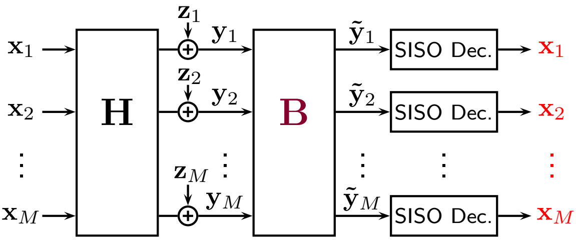
- Decode in order π. Cancel $\mathbf{x}_{\pi(1)}, \ldots, \mathbf{x}_{\pi(m-1)}$ from $\tilde{\mathbf{y}}_m$:

$$\mathsf{SINR}_{\mathsf{SIC},\pi(m)}(\mathbf{H}) = \max_{\mathbf{b}_m} \frac{\mathsf{SNR}\ \|\mathbf{b}_m^{\mathsf{T}} \mathbf{h}_{\pi(m)}\|^2}{1 + \mathsf{SNR}\ \sum_{\ell=m+1}^{M} \|\mathbf{b}_m^{\mathsf{T}} \mathbf{h}_{\pi(\ell)}\|^2}$$

- Rate per user:

$$R_{\mathsf{V\text{-}BLAST\ II}}(\mathbf{H}) = \max_{\pi} \min_{m=1,\ldots,M} \log\left(1 + \mathsf{SINR}_{\mathsf{SIC},\pi(m)}(\mathbf{H})\right)$$

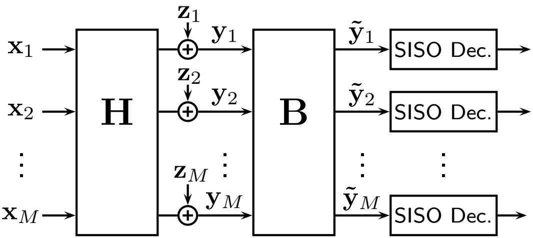
MIMO Uplink Channel: Integer-Forcing



What if we could decode something else?

- Zero-Forcing / LMMSE: First, eliminate interference.
  Then, decode individual data streams.

MIMO Uplink Channel: Integer-Forcing



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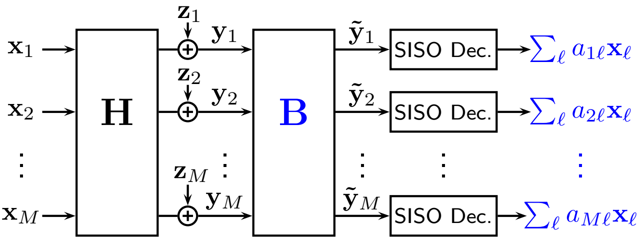
First, decode

MIMO Uplink Channel: Integer-Forcing



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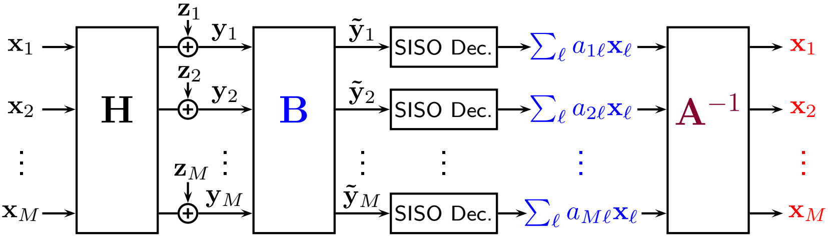
- Integer-Forcing: First, decode integer-linear combinations.



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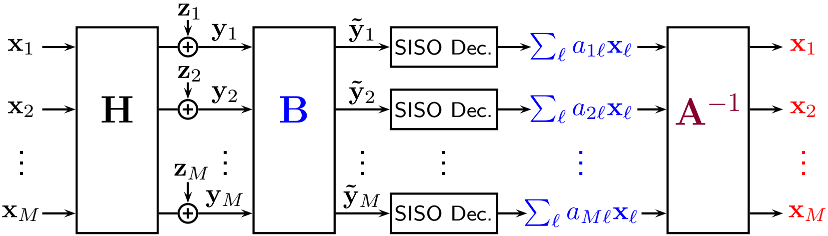
MIMO Uplink Channel: Integer-Forcing



What if we could decode something else?

- Zero-Forcing / LMMSE: First, eliminate interference.
  Then, decode individual data streams.

- Integer-Forcing: First, decode integer-linear combinations.
  Then, eliminate interference.

- If the integer matrix $\mathbf{A}$ is full rank, we can successfully recover the individual data streams.

Integer-Forcing Linear Receivers:

- The m^{th} effective channel after projection is

$$\mathbf{b}_m^{\mathsf{T}}\mathbf{Y} = \mathbf{b}_m^{\mathsf{T}}\mathbf{H}\mathbf{X} + \mathbf{b}_m^{\mathsf{T}}\mathbf{Z}$$

Integer-Forcing Linear Receivers:

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$$\begin{aligned}\mathbf{b}_m^{\mathsf{T}}\mathbf{Y} &= \mathbf{b}_m^{\mathsf{T}}\mathbf{H}\mathbf{X} + \mathbf{b}_m^{\mathsf{T}}\mathbf{Z} \\ &= \mathbf{a}_m^{\mathsf{T}}\mathbf{X} + (\mathbf{b}_m^{\mathsf{T}}\mathbf{H} - \mathbf{a}_m^{\mathsf{T}})\mathbf{X} + \mathbf{b}_m^{\mathsf{T}}\mathbf{Z}\end{aligned}$$

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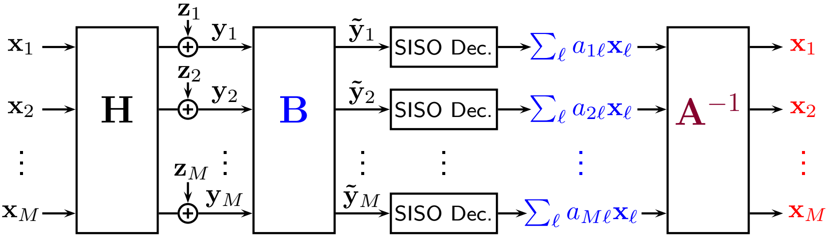
- The $a_{m\ell} \in \mathbb{Z}[j]$ are Gaussian integers and the codebook should be closed under integer-linear combinations.

- We are free to choose any full-rank integer-valued matrix $\mathbf{A}$.

MIMO Uplink Channel: Integer-Forcing



Integer-Forcing Linear Receivers: (**Zhan-Nazer-Erez-Gastpar '12**)

- The m^{th} SISO decoder tries to recover $\sum_\ell a_{m\ell}\mathbf{x}_\ell$ from $\mathbf{b}_m^\mathsf{T}\mathbf{Y}$:

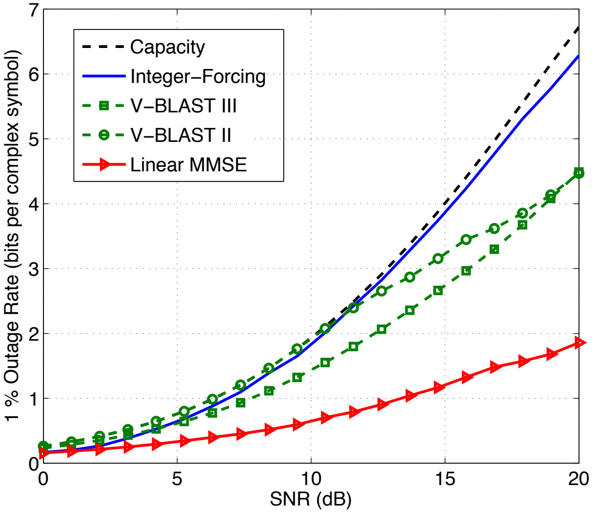
$$\mathsf{SINR}_{\mathsf{IF},m}(\mathbf{H},\mathbf{A}) = \max_{\mathbf{b}_m} \frac{\mathsf{SNR}}{\|\mathbf{b}_m\|^2 + \mathsf{SNR}\|\mathbf{b}_m^\mathsf{T}\mathbf{H} - \mathbf{a}_m^\mathsf{T}\|^2}$$

- Rate per user:

$$R_{\mathsf{IF}}(\mathbf{H}) = \max_{\mathbf{A}} \min_{m=1,\dots,M} \log^+\left(\mathsf{SINR}_{\mathsf{IF},m}(\mathbf{H},\mathbf{A})\right)$$

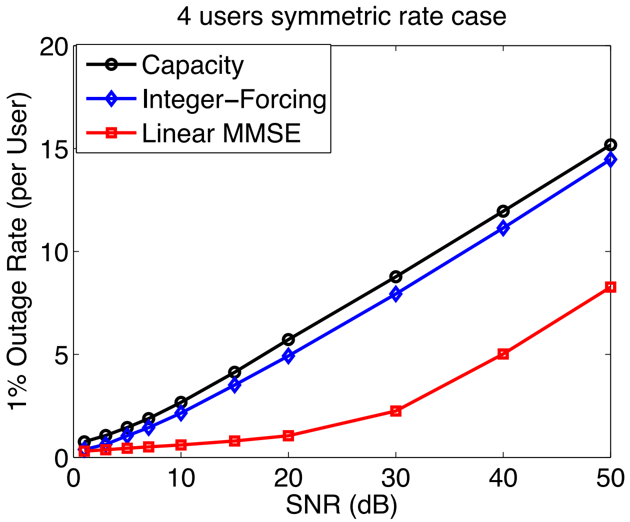
- Includes linear MMSE as a special case by setting $\mathbf{A} = \mathbf{I}$.

Comparison: Outage Rates



2 users, 2 receive antennas, Rayleigh fading, 1% outage.

Comparison: Outage Rates



4 users, 4 receive antennas, Rayleigh fading, 1% outage.

Questions

- How can we efficiently select a good integer matrix $\mathbf{A}$?
- How does the performance scale with the number of users?
- How sensitive is the performance to imperfect CSIR?
- What types of SISO encoders and decoders can we use?
- What about the downlink?
- Can we move beyond this idealized problem setting?

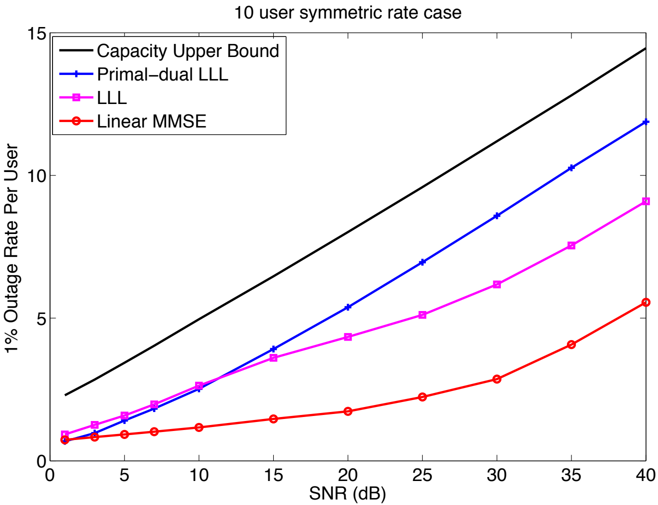
Finding a Good Integer Matrix

$$\mathsf{SINR}_{\mathsf{IF},m}(\mathbf{H},\mathbf{A}) = \max_{\mathbf{b}_m} \frac{\mathsf{SNR}}{\|\mathbf{b}_m\|^2 + \mathsf{SNR}\|\mathbf{H}^\mathsf{T}\mathbf{b}_m - \mathbf{a}_m\|^2}$$

- Optimal $\mathbf{b}_m$ is the MMSE projection.
- Plugging in and applying the Matrix Inversion Lemma, we get that

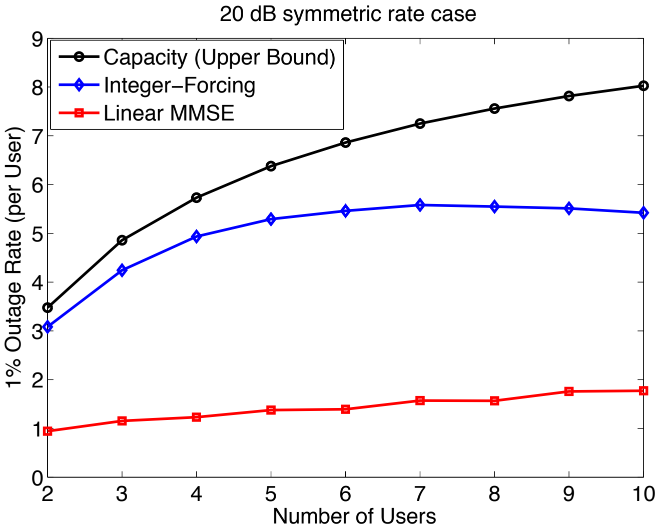
$$\mathsf{SINR}_{\mathsf{IF},m}(\mathbf{H},\mathbf{A}) = \frac{1}{\left\|\left(\mathbf{I} + \mathsf{SNR}\ \mathbf{H}^*\mathbf{H}\right)^{-1/2}\mathbf{a}_m\right\|^2}$$

- Finding the optimal $\mathbf{A}$ corresponds to finding a good lattice basis.
- This is a hard problem in general but good approximation algorithms are known, such as the LLL algorithm.
- We are currently using a slight twist: We run LLL to get a lattice basis. Then, we turn to the dual lattice and run LLL again, initializing with the first basis.



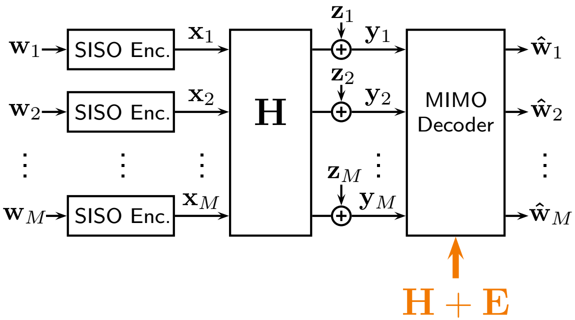
Rayleigh fading, 1% outage.

How does the performance scale with the number of users?



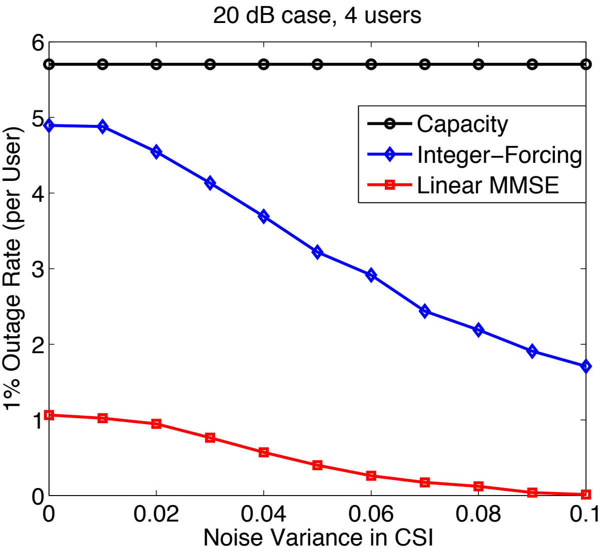
Rayleigh fading, 1% outage.

What is the impact of imperfect CSIR?



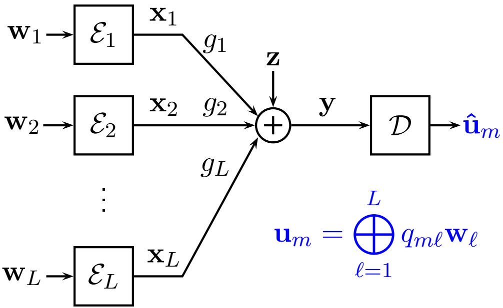
- Receiver only sees $\mathbf{H} + \mathbf{E}$ where $\mathbf{E}$ is elementwise i.i.d. $\mathcal{CN}(0, \sigma^2)$.
- May result in selecting both a suboptimal integer matrix $\mathbf{A}$ and a suboptimal projection matrix $\mathbf{B}$.

What is the impact of imperfect CSIR?



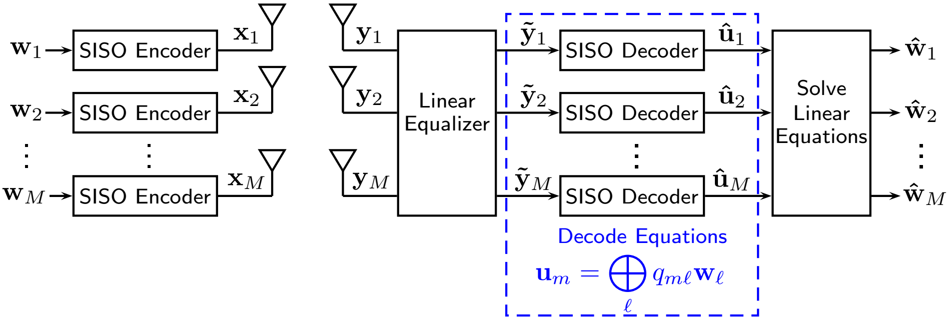
4 users, 20dB, Rayleigh fading, 1% outage.

What kinds of SISO coding schemes can be used?



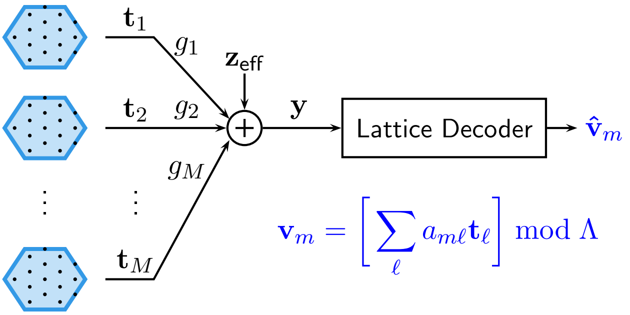
- Underlying integer-forcing is the compute-and-forward framework, which is used as a black box to recover linear combinations of the messages over some finite field $\mathbb{F}_p$.
- Messages are vectors over a prime-sized finite field, $\mathbf{w}_\ell \in \mathbb{F}_p^k$.

Integer-Forcing Linear Receiver



- Architecture is completely digital after SISO decoders.

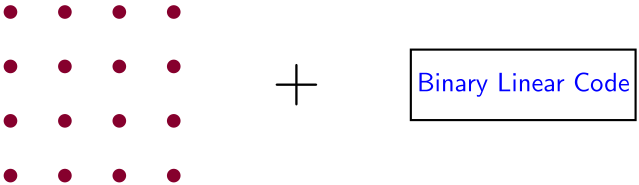
What kinds of SISO coding schemes can be used?



$$\mathbf{v}_m = \Big[\sum_{\ell} a_{m\ell} \mathbf{t}_\ell \Big] \bmod \Lambda$$

- **Nazer-Gastpar '11:** Compute-and-forward achievability proofs via nested lattice codes.
- High-dimensional nested lattice codes lead to nice $\log(\text{SINR})$ expressions but have high implementation complexity.
- Remember, all we actually need is that the codebook is closed under integer-linear combinations.

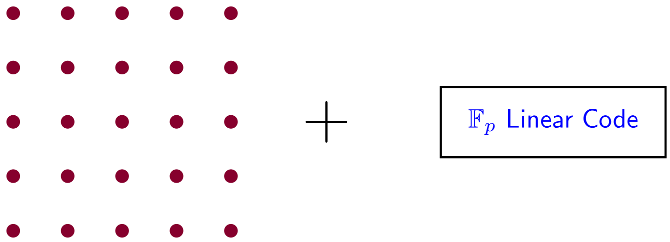
What kinds of SISO coding schemes can be used?



- What about QAM combined with a binary linear code?
- Issue: Real addition does not map well to addition over $\mathbb{F}_{2^M}$.

$$[x_1 + x_2] \bmod 2^M \neq x_1 \oplus x_2$$

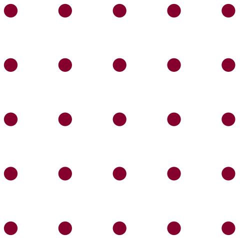
What kinds of SISO coding schemes can be used?



- What about p-ary QAM where p is prime combined with a linear code over $\mathbb{F}_p$?
- Real addition maps well to addition over $\mathbb{F}_p$.

$$[x_1 + x_2] \bmod p = x_1 \oplus x_2$$

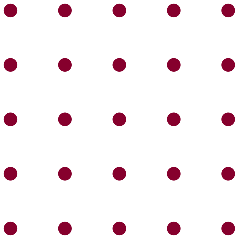
Uncoded Integer-Forcing



Uncoded Integer-Forcing:

- Project by $\mathbf{b}_m$, take $\bmod\ p$, apply slicer.
- Correct if we recover $[a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mM}x_M] \bmod p$ for all m.

Uncoded Integer-Forcing



Uncoded Integer-Forcing:

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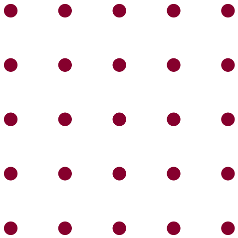
- Is this lattice-aided reduction?

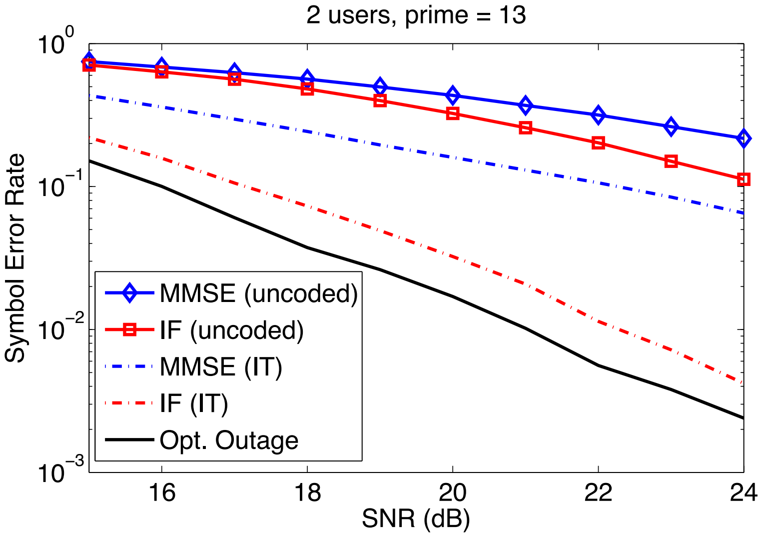
Uncoded Integer-Forcing



Uncoded Integer-Forcing:

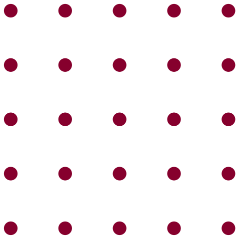
- Project by $\mathbf{b}_m$, take $\bmod\ p$, apply slicer.
- Correct if we recover $[a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mM}x_M] \bmod p$ for all m.
- Is this lattice-aided reduction? Nearly. We add the $\bmod\ p$.

Uncoded Integer-Forcing



2 users, 2 receive antennas, Rayleigh fading

$p = 13$, fixed rate $1/2 \log(13)$.

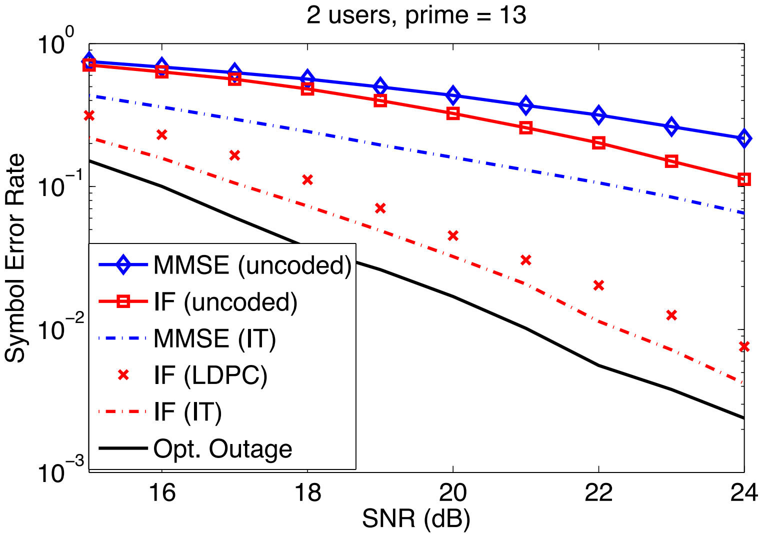


$+$

LDPC Code over $\mathbb{F}_p$

Coded Integer-Forcing:

- Project by $\mathbf{b}_m$, take $\mod p$, apply LDPC decoding algorithm.
- Correct if we recover $[a_{m1}\mathbf{x}_1 + a_{m2}\mathbf{x}_2 + \cdots + a_{mM}\mathbf{x}_M] \mod p$ for all m.



2 users, 2 receive antennas, Rayleigh fading

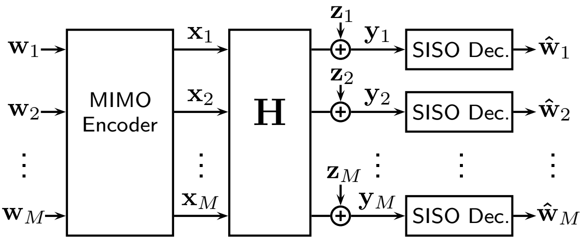
$p = 13$, fixed rate $1/2 \log(13)$, regular $(3, 6)$ LDPC code.

- Lots of interesting questions on how to design low-complexity constellations and linear codes that work well for compute-and-forward.
- Several recent papers and...

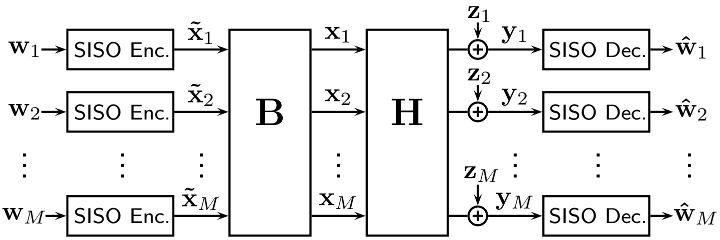
Codes for Compute-and-Forward

- Lots of interesting questions on how to design low-complexity constellations and linear codes that work well for compute-and-forward.
- Several recent papers and...
- Krishna's talk coming up next!

MIMO Downlink Channel



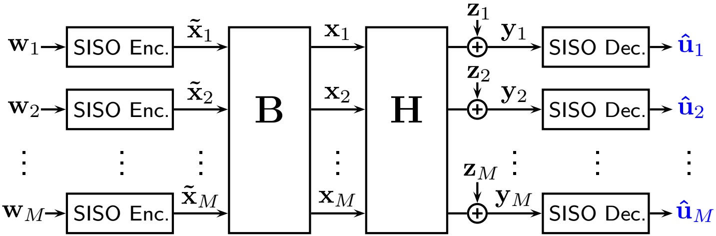
- Capacity region is known. Requires dirty-paper coding.
  Caire-Shamai '03, Vishwanath-Jindal-Goldsmith '04,
  Viswanath-Tse '03, Yu-Cioffi '04, Weingarten-Steinberg-Shamai '06.



Zero-Forcing Beamforming:

- Use beamforming matrix $\mathbf{B}$ to eliminate interference between data streams.

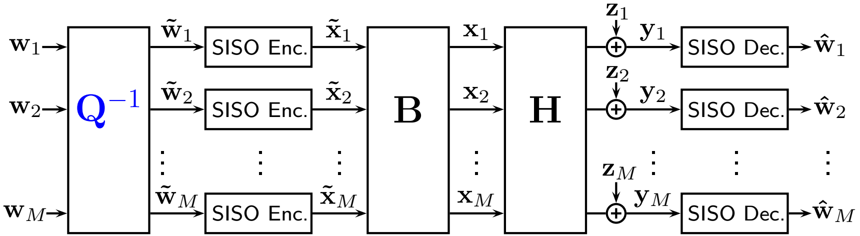
MIMO Downlink Channel: Integer-Forcing



$$\mathbf{u}_m = \bigoplus_{\ell} q_{m\ell} \mathbf{w}_\ell$$

Integer-Forcing Beamforming: (**Hong-Caire '12,'13**)

- Use beamforming matrix $\mathbf{B}$ to create an integer-valued effective channel $\mathbf{A}$.
- Decode linear combinations with $q_{m\ell} = [a_{m\ell}] \bmod p$.

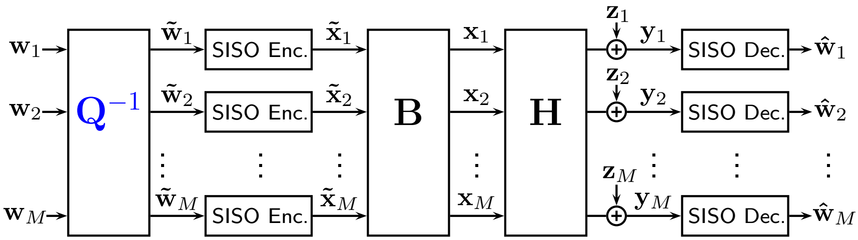


Integer-Forcing Beamforming: (**Hong-Caire '12,'13**)

- Use beamforming matrix $\mathbf{B}$ to create an integer-valued effective channel $\mathbf{A}$.

- ~~Decode linear combinations with $q_{m\ell} = [a_{m\ell}] \bmod p$.~~
  Pre-invert $\mathbf{Q} = [\mathbf{A}] \bmod p$ and decode messages.

MIMO Downlink Channel: Integer-Forcing



Integer-Forcing Beamforming: (**Hong-Caire '12,'13**)

- Use beamforming matrix $\mathbf{B}$ to create an integer-valued effective channel $\mathbf{A}$.
- ~~Decode linear combinations with $q_{m\ell} = [a_{m\ell}] \bmod p$.~~ Pre-invert $\mathbf{Q} = [\mathbf{A}] \bmod p$ and decode messages.
- In very recent work, we have shown that uplink-downlink duality holds for integer-forcing. **He-Nazer-Shamai '14**

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- **Ordentlich-Erez-Nazer '13:** Framework for IF-SIC. Exact optimality if CSIT is available. Rate points tend to lie very close to the symmetric capacity.

Key Issues Going Forward

- Low-complexity constellations and codes.
- New algorithms for finding integer matrix $\mathbf{A}$.
- Synchronization.
- What if the channel realization changes over the coding blocklength? (e.g., OFDM)
- How should we include rate adaptation?
- What does this mean for user selection?
- With Behnaam, Krishna, and students, we are working towards a WARP implementation.
- Any others?