The Approximate Capacity Region of the Phase-Fading, Very Strong, K-User Gaussian Interference Channel

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Abstract

Consider the K-user Gaussian interference channel with static channel coefficients. Recent efforts have characterized the capacity region to within a constant gap for several important special cases, including two-user, very weak, many-to-one, and one-to-many interference channels. The general case remains open, primarily due to the challenge of characterizing the achievable rate for interference alignment strategies outside of the very high signal-to-noise ratio (SNR) limit. This paper argues that some of these difficulties can be overcome if the phases of the channel coefficients are time-varying. The main result is a constant-gap approximation of the capacity region of phase-fading, Gaussian K-user interference channels where all receivers are in the very strong interference regime.

I. Introduction

The capacity region of the K-user Gaussian interference channel is a long-standing open problem. While for $K = 2$ users the capacity region is now known to within one bit [1], the $K > 2$ case has proven considerably more challenging, due to the possibility of interference alignment [2], [3]. To date, the approximate capacity is only known for a few special cases that require interference alignment, such as many-to-one [46], one-to-many [46], and symmetric channels [49], [55]. The aim of this paper is to show that progress can be made by relaxing the assumption that the channel gains are static, and instead allowing the phases to vary with time while holding the magnitudes fixed.

It is well-known that time-varying (i.e., parallel) Gaussian point-to-point [7], [8], multiple-access [8]–[10], and broadcast [8], [11]–[13] channels are *separable*. That is, the capacity region can be attained by separately coding across each channel realization combined with power allocation (i.e., waterfilling). On the other hand, time-varying Gaussian interference channels are *inseparable* [14]–[16], meaning that the capacity region can only be attained by jointly coding across channel realizations. At a first glance, it might appear that inseparability increases the

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difficulty of characterizing the capacity region. However, for the interference channel, coding across many channel realizations often leads to simpler achievable strategies and rate expressions.

Our coding strategy combines ideas from the Cadambe-Jafar alignment scheme for time-varying channels [3], ergodic interference alignment [17], and compute-and-forward [18]. Recall that the ergodic alignment scheme pairs up time slots such that the sum of the two paired channel matrices yields a diagonal matrix, corresponding to no interference. By repeating each codeword symbol over paired time slots, it is possible for all users to simultaneously operate at half their respective single-user capacities. As shown by Jafar [19], ergodic alignment achieves the capacity region of phase-fading interference channels if the channel is in a bottleneck state, i.e., each receiver sees an interferer with equal strength to its desired signal. Here, we examine the scenario where each interferer is in the very strong regime with respect to every receiver according to the two-user criterion established by Carleial [20]. In this setting, ergodic alignment alone does not suffice.

The goal of our coding strategy is to align the interference at each receiver into a single effective interfering codeword in order to allow each user to operate as if it faced a single very strong interferer. To accomplish this objective, all transmitters employ the same lattice codebook, which guarantees that the integer-linear combination of codewords are themselves codewords. Of course, the time-varying channel gains are not integer-valued, and the transmitters must use careful power control in order to align lattice codewords from the perspective of the receiver. However, simultaneously modifying all of the interfering channel gains to have integer values is an overconstrained problem. To resolve this issue, we use a variation of the Cadambe-Jafar alignment scheme [3] across many channel realizations, combined with careful power control.

A. *Related Work*

Considerable progress was made on the capacity region of the two-user interference channel in the 1970s and 1980s, beginning with the inner and outer bounds of Ahlswede [21], Sato [22], and Carleial [23]. The capacity region of the Gaussian two-user interference channel was characterized in the very strong regime by Carleial [20] and in the strong regime by Sato [24] and Han and Kobayashi [25], with the latter also proposing an achievable rate region for all regimes. Costa and El Gamal generalized this strong regime capacity result from Gaussian to discrete memoryless channels [26] and also determined the full capacity for a class of semi-deterministic interference channels [27].

The 2000s saw a renewed interest in the two-user Gaussian interference channel, beginning with the one-bit capacity region approximation due to Etkin *et al.* [1]. Afterwards, three groups [28]–[30] independently and concurrently discovered that, when the interference is sufficiently weak, treating interference as noise is optimal. Subsequent efforts yielded constant-gap capacity approximations for several two-user Gaussian generalizations including compound channels [31], channel output feedback [32], transmitter cooperation [33], [34], receiver cooperation [35], and multiple antennas [36]. Recent work [37] has also settled the Costa corner-point conjecture [38], [39].

The discovery of interference alignment by Motahari *et al.* [2] and Cadambe and Jafar [3] sparked several efforts to characterize the K-user capacity region. Using variations on the linear Cadambe-Jafar scheme, the degrees-of-freedom region has been characterized for a large class of time-varying (or frequency-selective) interference networks [40]–[44]. There is a rich body of work on linear alignment, and we refer interested readers to [45] for a survey.

For static channel coefficients, linear interference alignment often corresponds to an overconstrained system of linear equations. However, as first noted by Bresler *et al.* [46] for many-to-one interference channels, it is possible to align interference on the *signal scale* through the use of lattice codebooks. Lattice alignment was subsequently used by Motahari *et al.* [47] to show that $K/2$ degrees-of-freedom are achievable up to a set of channel matrices with measure zero, i.e., the degrees-of-freedom are discontinuous with respect to the channel gains [4]–[6]. A similar phenomenon appears in the best known approximate capacity characterizations for the two-user X channel [48] and K-user symmetric interference channel [49], and an "outage set" is needed as part of the characterization. We also note that, for K-user interference channels, Geng *et al.* [50] have developed sufficient conditions on when treating interference as noise approximates the capacity to within a constant gap [50].

Ergodic alignment has been used as a building block to develop achievable strategies for dense interference networks [19], [51], multi-hop interference networks [52], [53], and multiple-access wiretap channels [54]. Our proof strategy closely resembles the computation alignment scheme proposed by Niesen *et al.* [53] for approximating the capacity of time-varying multi-hop interference networks.

B. Paper Organization

The rest of the paper is organized as follows. In Section II, we give a formal problem statement. In Section III, we summarize previous work on the very strong interference regime, state our main result, and provide a motivating example. Section IV describes the channel quantization process used in the ergodic alignment scheme. Finally, we present our achievable strategy for $K = 3$ users in Section V and for $K > 3$ users in Section VI.

II. Problem Statement

We begin by describing our notation. We will denote column vectors with lowercase boldface letters (e.g., $\mathbf{x} \in \mathbb{C}^K$) and matrices using boldface uppercase letters (e.g., $\mathbf{H} \in \mathbb{C}^{K\times K}$). Let $\mathbf{x}^\mathsf{T}$ and $\mathbf{x}^*$ denote the transpose and conjugate transpose of $\mathbf{x}$, respectively. Let $\mathrm{diag}(\mathbf{x})$ as well as $\mathrm{diag}(\mathbf{x}^\mathsf{T})$ denote the diagonal matrix formed by placing the elements of $\mathbf{x}$ along the diagonal. For a real scalar x, let $\lceil x \rceil$ denote its ceiling (i.e., the smallest integer greater than or equal to x), $\lfloor x \rfloor$ denote its floor (i.e., the largest integer less than or equal to x), and $\lfloor x \rceil$ denote the nearest integer (with ties rounded up). For a complex scalar x, let $|x|$ denote its magnitude and $\measuredangle x$ denotes its phase in radians. For a complex vector $\mathbf{x}$, let $\|\mathbf{x}\|$ denote its Euclidean norm. All logarithms are taken with respect to base-2 and we define $\log^+(x) = \max(\log(x), 0)$. Define $\mathbf{I}$ to be the identity matrix and $\mathbf{1}$ to be the all-ones vector.

We now provide necessary definitions for a time-varying K-user Gaussian interference channel. See Figure 1 for a block diagram.

Definition 1 (Messages): Each transmitter (indexed by $\ell = 1, \ldots, K$) has a message m_ℓ that is generated independently and uniformly over $\{1, 2, \ldots, 2^{TR_\ell}\}$ for some rate $R_\ell \geq 0$.

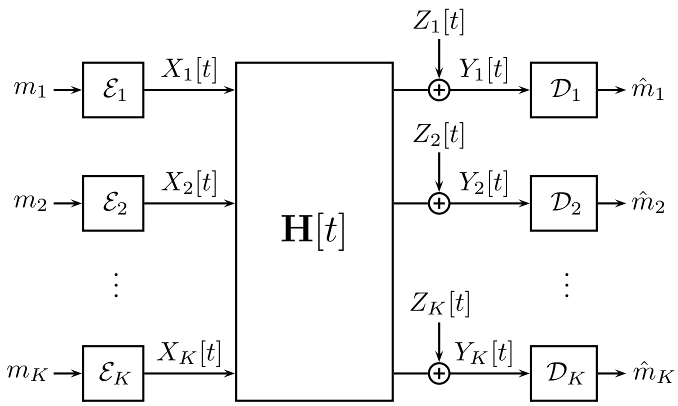

Fig. 1. K-user Gaussian interference channel with time-varying channel coefficients.

Definition 2 (Encoders): Each transmitter has an encoding function $\mathcal{E}_\ell : \{1, 2, \ldots, 2^{TR_\ell}\} \to \mathbb{C}^T$ that maps its message m_ℓ into a sequence of T complex-valued channel inputs $x_\ell[1], \ldots, x_\ell[T]$ satisfying an average power constraint

$$\frac{1}{T}\sum_{t=1}^{T} |x_\ell[t]|^2 \leq P \ .$$

Definition 3 (Channel Model): The channel output at time t at receiver k is

$$y_k[t] = \sum_{\ell=1}^{K} h_{k,\ell}[t] x_\ell[t] + z_k[t] \tag{1}$$

where $h_{k,\ell}[t] \in \mathbb{C}$ denotes the channel gain from transmitter ℓ to receiver k at time t. Let $\mathbf{H}[t] = \{h_{k,\ell}[t]\}$ denote the channel matrix at time t. We will assume full channel state information (CSI) is available, i.e., at time t, all transmitters and receivers know $\mathbf{H}[t]$.

We will focus on the important special case where the magnitudes of the channel gains are constant across time[1], i.e., $|h_{k,\ell}[t_1]| = |h_{k,\ell}[t_2]|$ for all t_1 and t_2. For notational convenience, we will drop the time index when referring to the magnitudes of the channel gains, i.e. instead of writing $|h_{k,\ell}[t]|$, we will write $|h_{k,\ell}|$. Furthermore, we assume that the phase of each channel gain $\measuredangle h[t]$ is i.i.d.[2] across time according to a uniform distribution over $[0, 2\pi)$.

The noise $z_k[t]$ is i.i.d. across time according to a circularly symmetric complex Gaussian with mean zero and unit variance, $z_k[t] \sim \mathcal{CN}(0, 1)$, and is independent of all channel inputs and channel gains.

Definition 4 (Decoders): Each receiver (indexed by $k = 1, \ldots, K$) makes an estimate $\hat{m}_k$ of its desired message m_k using a decoding function $\mathcal{D}_k : \mathbb{C}^T \to \{1, 2, \ldots, 2^{TR_k}\}$.

[1]Without this assumption, it becomes more challenging to parameterize the very strong regime, due to the inseparability of parallel interference channels. See [14] for more details.

[2]For our achievable scheme, it suffices to assume that each phase sequence is stationary and ergodic, and all sequences are independent of one another. However, the i.i.d. assumption will help simplify our proofs. See [53, §IV] for a sample proof for the stationary ergodic case.

Definition 5 (Achievable Rates): A rate tuple $(R_1, \ldots, R_K)$ is achievable if, for any $\epsilon > 0$ and T large enough, there exist encoders and decoders that can attain probability of error at most ϵ,

$$\mathbb{P}\left(\bigcup_{k=1}^{K}\{\hat{m}_k \neq m_k\}\right) < \epsilon.$$

Definition 6 (Capacity Region): The capacity region $\mathscr{C}$ is the closure of the set of all achievable rate tuples.

III. Main Result

Below, we begin with a brief overview of known capacity results for *static* very strong interference channels. Afterwards, we will state our constant-gap results for *time-varying* interference channels. Finally, we provide an example to illustrate some of the key ideas underlying our coding scheme.

A. Static Very Strong Interference Channels

Consider the static, two-user Gaussian interference channel with channel outputs

$$y_1[t] = h_{1,1}x_1[t] + h_{1,2}x_2[t] + z_1[t] \tag{2}$$

$$y_2[t] = h_{2,1}x_1[t] + h_{2,2}x_2[t] + z_2[t]\ . \tag{3}$$

We say that the channel is in the very strong regime if

$$|h_{1,2}|^2 \geq |h_{2,2}|^2(1 + |h_{1,1}|^2 P) \tag{4}$$

$$|h_{2,1}|^2 \geq |h_{1,1}|^2(1 + |h_{2,2}|^2 P)\ . \tag{5}$$

As shown by Carleial [20], the capacity region in the very strong regime is the set of rate pairs satisfying

$$R_1 \leq \log(1 + |h_{1,1}|^2 P) \tag{6}$$

$$R_2 \leq \log(1 + |h_{2,2}|^2 P)\ , \tag{7}$$

i.e., both users can simultaneously operate at their interference-free capacities. The achievable strategy is for the transmitters to employ i.i.d. Gaussian codebooks. Each receiver decodes the interference first while treating its desired codeword as noise. It then cancels out the interference and proceeds to decode its desired codeword.

The same strategy can be applied for $K > 2$ users. However, since each receiver recovers all $K - 1$ interfering codewords prior to recovering its desired codeword, the interfering channel gains must be much stronger in order for all users to reach their interference-free capacities. Lattice interference alignment offers the possibility of relaxing these conditions by allowing each receiver to directly decode and cancel the *sum* of the interfering codewords.

Consider the static, symmetric Gaussian interference channel with channel output

$$y_k[t] = x_k[t] + h\sum_{\ell \neq k} x_\ell[t] + z_k[t] \tag{8}$$

where $h \in \mathbb{C}$ is the symmetric cross-channel gain. If all users employ the same lattice codebook, the sum of interfering codewords will correspond to a single effective codeword,

$$\mathbf{x}_{\text{int},k} = \sum_{\ell \neq k} \mathbf{x}_\ell \tag{9}$$

where $\mathbf{x}_\ell = [x_\ell[1] \ \cdots \ x_\ell[T]]^\mathsf{T}$ is the ℓ^{th} codeword. As shown by Sridharan *et al.* [55], the receiver can decode the sum of interfering codewords if the rates satisfy

$$R_\ell < \log\left(\frac{|h|^2 P}{1+P}\right). \tag{10}$$

Thus, if

$$|h|^2 \geq \frac{(1+P)^2}{P} \ , \tag{11}$$

then each user can operate at its interference-free capacity $\log(1+P)$. Note that this is a substantial improvement on the condition for reaching the interference-free capacity while decoding the $K-1$ interfering codewords individually,

$$|h|^2 \geq \frac{(1+P)\big((1+P)^{K-1}-1\big)}{(K-1)P} \ . \tag{12}$$

This alignment strategy does not immediately generalize beyond the symmetric case, since it requires that each receiver observes an integer-linear combination of the interfering codewords.

B. Phase-Fading, Very Strong Interference Channels

Our main result is that, for interference channels with time-varying phase, if all receivers are in the very strong interference regime, the capacity region can be determined to within a constant gap. We define the very strong interference regime by applying Carleial's condition [20] to each pair of desired and interfering channel gains. Specifically, we assume that

$$|h_{k,\ell}|^2 \geq |h_{\ell,\ell}|^2(1+|h_{k,k}|^2 P) \quad \forall \ell \neq k \ . \tag{13}$$

Theorem 1: Consider a very strong, phase-fading Gaussian interference channel. Any rate tuple $(R_1, \dots, R_K)$ satisfying

$$R_\ell < \log(1+|h_{\ell,\ell}|^2 P) - 2K^2 - 1 \tag{14}$$

is achievable.

The achievable strategy combines ideas from ergodic alignment [17], [53] and lattice-based interference alignment [46], [49], [55]. The proof is split into two pieces: Section V establishes the lower bound for $K = 3$ users and Section VI generalizes this result to $K > 3$ users.

Note that, since each user cannot exceed its interference-free capacity, Theorem 1 characterizes the very strong capacity region to within $2K^2 - 1$ bits per user.

C. Motivating Example

We now illustrate our alignment strategy with a small example for $K = 3$ users. Consider a time slot t_1 and the channel gains $h_{k,\ell}[t_1]$. With a slight abuse of notation, we will drop the time index and use $h_{k,\ell}$ to denote $h_{k,\ell}[t_1]$. Hence, we have

$$\mathbf{H}[t_1] = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} \\ h_{2,1} & h_{2,2} & h_{2,3} \\ h_{3,1} & h_{3,2} & h_{3,3} \end{bmatrix} \ . \tag{15}$$

We would like to induce lattice alignment at all three receivers using a linear strategy. This is an overconstrained problem if we limit ourselves to one channel matrix. Instead, we will select a second time slot t_2 whose channel matrix satisfies

$$\mathbf{H}[t_2] = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} \\ h_{2,1} & h_{2,2} & -h_{2,3} \\ h_{3,1} & h_{3,2} & h_{3,3} \end{bmatrix} . \tag{16}$$

Of course, the probability of such a match occurring within a finite number of time slots is zero. Later, we will carefully argue that, if we allow for a slight mismatch, we can operate at nearly the same rates with a finite blocklength.

We assume that transmitter 1 has two symbols $s_{1,1}$ and $s_{1,2}$, transmitter 2 has two symbols $s_{2,1}$ and $s_{2,2}$, and transmitter 3 has one symbol $s_{3,1}$, each with average power P. Let $\mathbf{x}_\ell = \big[x_\ell[t_1] \ x_\ell[t_2]\big]^{\mathsf{T}}$ be the vector of channel inputs from transmitter ℓ. The first channel input is a weighted sum of the symbols and the second is a weighted difference,

$$\mathbf{x}_1 = \begin{bmatrix} \beta_{1,1}s_{1,1} + \beta_{1,2}s_{1,2} \\ \beta_{1,1}s_{1,1} - \beta_{1,2}s_{1,2} \end{bmatrix} \tag{17}$$

$$\mathbf{x}_2 = \begin{bmatrix} \beta_{2,1}s_{2,1} + \beta_{2,2}s_{2,2} \\ \beta_{2,1}s_{2,1} - \beta_{2,2}s_{2,2} \end{bmatrix} \tag{18}$$

$$\mathbf{x}_3 = \begin{bmatrix} \beta_{3,1}s_{3,1} \\ \beta_{3,1}s_{3,1} \end{bmatrix} \tag{19}$$

where the scaling coefficients $\beta_{\ell,n}$ will be chosen such that the interfering symbols at each receiver form an integer-linear combinations. The magnitudes are chosen to satisfy $\frac{1}{2} \leq |\beta_{\ell,n}| \leq 1$ so that

- The power constraint is never violated.
- In the worst case, the effective power of each symbol is reduced by a factor of $1/4$, corresponding to a rate loss of 2 bits.

Let $\mathbf{y}_k = \big[y_k[t_1] \ y_k[t_2]\big]^{\mathsf{T}}$ be the vector of channel outputs at receiver k. To extract its message, each receiver first takes the sum and difference of its observations to obtain the effective channel output vector $\tilde{\mathbf{y}}_k = [\tilde{y}_{k,1} \ \tilde{y}_{k,2}]^{\mathsf{T}}$ where

$$\tilde{y}_{k,1} = \frac{1}{2}\big(y_k[t_1] + y_k[t_2]\big) \qquad \tilde{y}_{k,2} = \frac{1}{2}\big(y_k[t_1] - y_k[t_2]\big) \ .$$

Combining terms, we get

$$\tilde{\mathbf{y}}_1 = \begin{bmatrix} h_{1,1}\beta_{1,1}s_{1,1} + h_{1,2}\beta_{2,1}s_{2,1} + h_{1,3}\beta_{3,1}s_{3,1} \\ h_{1,1}\beta_{1,2}s_{1,2} + h_{1,2}\beta_{2,2}s_{2,2} \end{bmatrix} + \tilde{\mathbf{z}}_1$$

$$\tilde{\mathbf{y}}_2 = \begin{bmatrix} h_{2,1}\beta_{1,1}s_{1,1} + h_{2,2}\beta_{2,1}s_{2,1} \\ h_{2,1}\beta_{1,2}s_{1,2} + h_{2,2}\beta_{2,2}s_{2,2} + h_{2,3}\beta_{3,1}s_{3,1} \end{bmatrix} + \tilde{\mathbf{z}}_2$$

$$\tilde{\mathbf{y}}_3 = \begin{bmatrix} h_{3,1}\beta_{1,1}s_{1,1} + h_{3,2}\beta_{2,1}s_{2,1} + h_{3,3}\beta_{3,1}s_{3,1} \\ h_{3,1}\beta_{1,2}s_{1,2} + h_{3,2}\beta_{2,2}s_{2,2} \end{bmatrix} + \tilde{\mathbf{z}}_3$$

where $\tilde{\mathbf{z}}_k = [\tilde{z}_{k,1} \ \tilde{z}_{k,2}]^\mathsf{T}$ and

$$\tilde{z}_{k,1} = \frac{1}{2}\big(z_k[t_1] + z_k[t_2]\big) \qquad \tilde{z}_{k,2} = \frac{1}{2}\big(z_k[t_1] - z_k[t_2]\big) \ .$$

Notice that, due to the sign flip for $h_{2,3}[t_2]$ in (16), the symbol $s_{3,1}$ appears in the second effective channel output at receiver 2, rather than the first. This will give our linear scheme the flexibility needed to align interference at all receiver.

Let h be a complex scalar. The following basic quantization properties will be used frequently in our proof:

- If $|h| \geq 1$, then $\frac{1}{2} \leq \frac{\lfloor |h| \rfloor}{|h|} \leq 1$.
- If $|h| \geq \frac{1}{2}$, then $\frac{1}{2} \leq \frac{|h|}{\lceil |h| \rceil} \leq 1$.

We now proceed to set the scaling coefficients, beginning with setting $\beta_{1,2} = 1$ and aligning interference for $\tilde{y}_{2,2}$. If $|h_{2,3}| \geq |h_{2,1}|$, we choose the integer coefficient and scaling parameter as follows:

$$b_2 = \left\lfloor \frac{|h_{2,3}|}{|h_{2,1}|} \right\rfloor \qquad \beta_{3,1} = \frac{b_2 h_{2,1}}{h_{2,3}}. \tag{20}$$

The resulting effective channel consists of the desired symbol plus an integer-linear combination of two interfering symbols plus noise,

$$\tilde{y}_{2,2} = h_{2,2}\beta_{2,2}s_{2,2} + h_{2,1}\left(s_{1,2} + b_2 s_{3,1}\right) + \tilde{z}_{2,2}. \tag{21}$$

On the other hand, if $|h_{2,3}| < |h_{2,1}|$, we choose

$$b_2 = \left\lceil \frac{|h_{2,1}|}{|h_{2,3}|} \right\rceil \qquad \beta_{3,1} = \frac{h_{2,1}}{b_2 h_{2,3}} \tag{22}$$

to obtain

$$\tilde{y}_{2,2} = h_{2,2}\beta_{2,2}s_{2,2} + h_{2,3}\beta_{3,1}\left(b_2 s_{1,2} + s_{3,1}\right) + \tilde{z}_{2,2}. \tag{23}$$

Now, we choose $\beta_{2,1}$ as a function of $\beta_{3,1}$ in order to align interference for $\tilde{y}_{1,1}$. Again, we carefully distinguish two cases:

- If $|h_{1,2}| \geq |h_{1,3}||\beta_{3,1}|$, the integer coefficient, scaling parameter, and resulting effective channel are

$$b_1 = \left\lfloor \frac{|h_{1,2}|}{|h_{1,3}||\beta_{3,1}|} \right\rfloor \qquad \beta_{2,1} = \frac{b_1 h_{1,3}\beta_{3,1}}{h_{1,2}}$$
$$\tilde{y}_{1,1} = h_{1,1}\beta_{1,1}s_{1,1} + h_{1,3}\beta_{3,1}(b_1 s_{2,1} + s_{3,1}) + \tilde{z}_{1,1}.$$

- Otherwise, if $|h_{1,2}| < |h_{1,3}||\beta_{3,1}|$, we set

$$b_1 = \left\lceil \frac{|h_{1,3}||\beta_{3,1}|}{|h_{1,2}|} \right\rceil \qquad \beta_{2,1} = \frac{h_{1,3}\beta_{3,1}}{b_1 h_{1,2}}$$
$$\tilde{y}_{1,1} = h_{1,1}\beta_{1,1}s_{1,1} + h_{1,2}\beta_{2,1}(s_{2,1} + b_1 s_{3,1}) + \tilde{z}_{1,1}.$$

Next, we set $\beta_{1,1}$ to align interference for $\tilde{y}_{3,1}$ as follows:

- If $|h_{3,1}| \geq |h_{3,2}||\beta_{2,1}|$, the integer coefficient, scaling parameter, and resulting effective channel are

$$b_3 = \left\lfloor \frac{|h_{3,1}|}{|h_{3,2}||\beta_{2,1}|} \right\rfloor \qquad \beta_{1,1} = \frac{b_3 h_{3,2}\beta_{2,1}}{h_{3,1}}$$
$$\tilde{y}_{3,1} = h_{3,3}\beta_{3,1}s_{3,1} + h_{3,2}\beta_{2,1}\left(b_3 s_{1,1} + s_{2,1}\right) + \tilde{z}_{3,1}.$$

- Otherwise, if $|h_{3,1}| < |h_{3,2}||\beta_{2,1}|$, we set

$$b_3 = \left\lceil \frac{|h_{3,2}||\beta_{2,1}|}{|h_{3,1}|} \right\rceil \qquad \beta_{1,1} = \frac{h_{3,2}\beta_{2,1}}{b_3 h_{3,1}}$$
$$\tilde{y}_{3,1} = h_{3,3}\beta_{3,1}s_{3,1} + h_{3,1}\beta_{1,1}\left(s_{1,1} + b_3 s_{2,1}\right) + \tilde{z}_{3,1}.$$

Finally, we note that there is only one interfering symbol in $\tilde{y}_{1,2}$ and no desired symbol in $\tilde{y}_{3,2}$. Thus, we are free to set $\beta_{2,2} = \beta_{3,2} = 1$. Overall, each desired symbol is observed over an effective channel output along with the integer-linear combination of (at most) two interfering symbols plus noise. Since this scheme sends 5 symbols over 2 channel uses, we will see a rate loss factor of $\frac{5}{6}$ per user (after time sharing). By repeating this strategy over many channel realizations and sending lattice codewords over the resulting effective channels, we can achieve the rates $R_\ell = \frac{5}{6}\log(1 + \frac{1}{4}|h_{\ell,\ell}|^2)$, where the factor of $1/4$ stems from the fact that $|\beta_{\ell,n}| \geq 1/2$. (See Section V-C for details on the lattice coding scheme.) In Section V, we will generalize this motivating example so that each user can send $(N-1)$ symbols over N channel uses, corresponding to a rate loss factor of $\frac{N-1}{N}$ per user. By taking $N \to \infty$, we can eliminate the rate loss factor entirely. Then, in Section VI, we will extend this scheme to $K > 3$ users.

IV. Channel Quantization

In order to establish a constant-gap result, our scheme carefully matches up channel matrices to create opportunities for lattice alignment. This matching process is carried out using a variation on the ergodic alignment technique from [17]. As noted earlier, the probability that a particular channel matrix occurs (corresponding to a perfect match) is zero. The key idea is to quantize the channel matrices into a finite number of bins, and then match up time slots based on quantized channel matrices. Overall, this allows us to match up nearly all channel matrices that occur within a large, finite blocklength.

We begin by noting that it suffices to quantize the channel phases since the magnitudes are fixed. We quantize the phase by dividing the interval of possible phases $[0, 2\pi)$ into $LN \max_{k,\ell} \lceil |h_{k,\ell}| \rceil$ equal-sized segments where $L, N \in \mathbb{N}$ are parameters to be chosen later. We represent each segment by its midpoint. Let $\mathcal{Q}$ denote the set of phase quantization points, $Q : [0, 2\pi) \to \mathcal{Q}$ the quantization function, and $\hat{h}_{k,\ell}[t] = |h_{k,\ell}| \exp(jQ(\measuredangle(h_{k,\ell}[t])))$ the quantized version of $h_{k,\ell}[t]$. This quantizer has the following useful properties:

- For any $h_{k,\ell}[t] \in \mathbb{C}$, the quantization error is at most

$$\left|\hat{h}_{k,\ell}[t] - h_{k,\ell}[t]\right| \leq \frac{\pi}{LN}. \tag{24}$$

- For any quantized channel gain $\hat{h}_{k,\ell} \in \mathcal{Q}$, multiplying by an N^{th} root of unity yields another quantization point,

$$\exp\left(\frac{j2\pi n}{N}\right)\hat{h}_{k,\ell} \in \mathcal{Q}, \tag{25}$$

  for all $n \in \mathbb{Z}$.

Denote the quantized channel matrix at time t by

$$\hat{\mathbf{H}}[t] = \{\hat{h}_{k,\ell}[t]\}_{k,\ell} \tag{26}$$

and let $\hat{\mathcal{H}}$ denote the set of all possible quantized channel matrices. By construction, $\hat{\mathcal{H}}$ is finite with size $|\hat{\mathcal{H}}| = \mathcal{Q}^{K^2}$.

The goal of the matching process is to create groups of N channel matrices that are amenable to lattice alignment. To this end, we will split the blocklength T into N consecutive subblocks, each of length T/N.[3] We will then choose T large enough so that each subblock is strongly typical. This will later allow us to match up all but a vanishing fraction of the channel matrices. The following lemma makes this precise.

Lemma 1: For any parameters $L, N \in \mathbb{N}$, and $\gamma > 0$, and for T large enough, we have that the empirical distribution of quantized channel matrices is close to the true distribution within all subblocks. Specifically, we have that

$$\sum_{t=(n-1)(T/N)+1}^{nT/N} \mathbb{1}\{\hat{\mathbf{H}}[t] = \hat{\mathsf{H}}\} \geq \frac{1-\gamma}{|\hat{\mathcal{H}}|}\frac{T}{N} \tag{27}$$

across all subblocks $n = 1, \dots, N$ and quantized channel realizations $\hat{\mathsf{H}} \in \hat{\mathcal{H}}$ with probability at least $1-\gamma$.

Proof: For a given $n \in \{1, \dots, N\}$, it follows from standard typicality arguments [56, §2.4] that, for T large enough, (27) is satisfied for all $\hat{\mathsf{H}} \in \hat{\mathcal{H}}$ with probability at least $1 - \frac{\gamma}{N}$. Using a union bound, we find that (27) holds for all n and $\hat{\mathsf{H}} \in \hat{\mathcal{H}}$ with probability at least $1-\gamma$. ∎

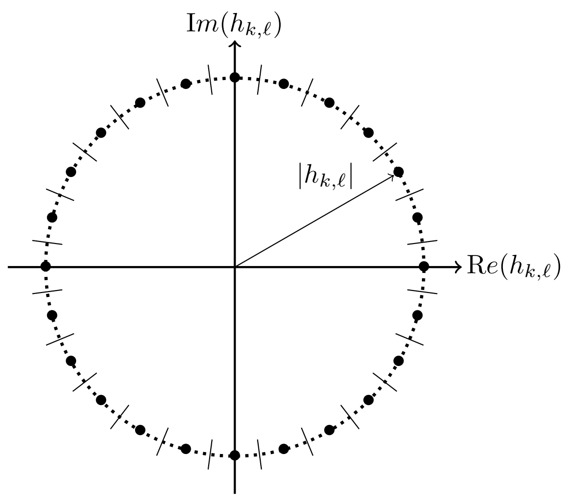

Fig. 2. Phase quantization for a single channel gain with $N = 4$, $L = 3$, and $\max_{k,\ell} |h_{k,\ell}| = 1.9$. The number of quantization bins is $LN \max_{k,\ell} \lceil |h_{k,\ell}| \rceil = 24$.

V. Proof of Theorem 1 for $K = 3$ Users

The alignment scheme of Cadambe and Jafar [3] admits a simpler form for $K = 3$ users. Since our coding strategy uses a variation on this scheme as a building block, it is simpler to start with the proof for $K = 3$ users. This will help us generate intuition for the $K > 3$ case. Our strategy consists of three components: channel matching, Fourier modulation and alignment, and lattice coding.

[3]Note that we have tacitly assumed that T is divisible by N and will do so throughout the paper.

A. Channel Matching

Our matching scheme groups together time slots whose channel matrices are well-suited for lattice alignment. The goal is to show that all but a vanishing fraction of time slots can be matched up.

We start by splitting the time slots into N consecutive subblocks, each of length T/N, with N to be specified later. Specifically, the n^{th} subblock refers to the time slots $t = (n-1)T/N + 1$ to $t = nT/N$. We employ the quantization scheme from Section IV for some $L \in \mathbb{N}$ to be specified later and the same N as above. For any $\gamma > 0$, we know from Lemma 1 that, for T large enough, the number of quantized channel realizations satisfies (27) for all subblocks $n = 1, \ldots, N$ and quantized channel realizations $\hat{\mathsf{H}} \in \hat{\mathcal{H}}$ with probability at least $1 - \gamma$. If, for some n and $\hat{\mathsf{H}}$, (27) does not hold, then we declare an error. For the remainder of this section, we condition on the event that (27) holds for all $n = 1, \ldots, N$ and $\hat{\mathsf{H}} \in \hat{\mathcal{H}}$.

For each subblock $n = 1, \ldots, N$ and quantized channel realization $\hat{\mathsf{H}} \in \hat{\mathcal{H}}$, we will designate the first $\frac{1-\gamma}{|\hat{\mathcal{H}}|} \frac{T}{N}$ time slots[4] time slots satisfying $\hat{\mathbf{H}}[t] = \hat{\mathsf{H}}$ for all $\hat{\mathsf{H}} \in \hat{\mathcal{H}}$ as useable, and ignore the rest. The total number of such useable time slots is $(1-\gamma)T$, which will impose a rate loss factor of $1 - \gamma$.

We will match each useable time slot in the first subblock with a unique time slot in each of the remaining $N - 1$ subblocks. Overall, this will produce exactly $(1-\gamma)T/N$ non-overlapping groups of N timeslots. Initialize $t_1 = 1$ and $B_{\hat{\mathsf{H}}} = 0$ for all $\hat{\mathsf{H}} \in \hat{\mathcal{H}}$, then run the following procedure:

1) Set $\hat{\mathsf{H}} = \hat{\mathbf{H}}[t_1]$. If $B_{\hat{\mathsf{H}}} < \frac{1-\gamma}{|\hat{\mathcal{H}}|} \frac{T}{N}$, then increment $B_{\hat{\mathsf{H}}}$ by 1 and go to Step 2. Otherwise, go to Step 4.
2) For each subblock $n = 2, \ldots, N$, find the first unused time slot t_n whose quantized channel matrix satisfies
$$\hat{\mathbf{H}}[t_n] = \begin{bmatrix} \hat{h}_{1,1}[t_1] & \hat{h}_{1,2}[t_1] & \hat{h}_{1,3}[t_1] \\ \hat{h}_{2,1}[t_1] & \hat{h}_{2,2}[t_1] & \omega^{-(n-1)}\hat{h}_{2,3}[t_1] \\ \hat{h}_{3,1}[t_1] & \hat{h}_{3,2}[t_1] & \hat{h}_{3,3}[t_1] \end{bmatrix} \tag{28}$$
where
$$\omega \triangleq \exp\Big(-\frac{j2\pi}{N}\Big). \tag{29}$$
3) Group the resulting time slots $t_1, \ldots, t_N$ together and mark them as "used" for the remainder of the matching procedure.
4) If $t_1 < T/N$, then increment t_1 by 1 and loop back to Step 1. Otherwise, terminate the procedure.

Recall that, by design, multiplying a quantization point by an N^{th} root of unity results in another quantization point. Since there are exactly $\frac{1-\gamma}{|\hat{\mathcal{H}}|} \frac{T}{N}$ useable time slots in each subblock with quantized channel realization $\hat{\mathsf{H}}$, then the procedure above places all useable time slots into matched groups.

Remark 1: Note that our scheme only requires instantaneous CSI at the transmitters. In other words, each transmitter can determine what to transmit at time t using only knowledge of $\mathbf{H}[t], \mathbf{H}[t-1], \ldots, \mathbf{H}[1]$ (or their quantized values).

[4]Since T will eventually be taken to infinity, we can safely assume that this quantity is integer-valued.

B. Fourier Modulation and Alignment

As discussed earlier, inducing lattice alignment over a single channel matrix with a linear strategy is infeasible. Our scheme takes N matched time slots and produces $N-1$ lattice-aligned effective channels at each receiver. The key idea is to exploit the phase offset for $\hat{h}_{2,3}[t_n]$ to change the effective channels over which the symbols from transmitter 3 symbols arrive at receiver 2. This symbol index shift provides the flexibility needed to induce lattice alignment, just as in the motivating example.

To transform the phase offsets into a symbol index shift, we will modulate our symbols using the Discrete Fourier Transform (DFT). Define the DFT matrix

$$\mathbf{W} \triangleq \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)^2} \end{bmatrix} \tag{30}$$

as well as the inverse DFT matrix

$$\mathbf{W}^{-1} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(N-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(N-1)} & \omega^{-2(N-1)} & \cdots & \omega^{-(N-1)^2} \end{bmatrix} \tag{31}$$

We also define the following matrix whose diagonal contains the N^{th} roots of unity,

$$\mathbf{F} \triangleq \operatorname{diag}\left(\begin{bmatrix} 1 \ \omega^{-1} \ \omega^{-2} \ \cdots \ \omega^{-(N-1)} \end{bmatrix}\right). \tag{32}$$

From the time-shifting property of the DFT [57, Table 3.7], it follows that

$$\mathbf{W}^{-1}\mathbf{F}\mathbf{W} \begin{bmatrix} s_1 \\ \vdots \\ s_{N-1} \\ s_N \end{bmatrix} = \begin{bmatrix} s_2 \\ \vdots \\ s_N \\ s_1 \end{bmatrix}. \tag{33}$$

We now describe our alignment scheme. Consider any group of matched time slots $t_1, \ldots, t_N$ and define

$$\mathbf{x}_\ell = \begin{bmatrix} x_\ell[t_1] \\ \vdots \\ x_\ell[t_N] \end{bmatrix} \quad \mathbf{y}_k = \begin{bmatrix} y_k[t_1] \\ \vdots \\ y_k[t_N] \end{bmatrix} \quad \mathbf{z}_k = \begin{bmatrix} z_k[t_1] \\ \vdots \\ z_k[t_N] \end{bmatrix} \tag{34}$$

$$\mathbf{D}_{k,\ell} = \operatorname{diag}\left(\begin{bmatrix} h_{k,\ell}[t_1], \ldots, h_{k,\ell}[t_N] \end{bmatrix}\right) \tag{35}$$

We can then concisely express the k^{th} channel output as

$$\mathbf{y}_k = \sum_{\ell=1}^{3} \mathbf{D}_{k,\ell}\mathbf{x}_\ell + \mathbf{z}_k.$$

It will be useful to express the channel outputs in terms of the quantized channel gains. Specifically, by defining

$$\hat{\mathbf{D}}_{k,\ell} = \mathrm{diag}\left(\left[\hat{h}_{k,\ell}[t_1] \ \cdots \ \hat{h}_{k,\ell}[t_N]\right]\right) \tag{36}$$

$$\hat{\mathbf{z}}_k = \sum_{\ell=1}^{3} (\mathbf{D}_{k,\ell} - \hat{\mathbf{D}}_{k,\ell})\mathbf{x}_k + \mathbf{z}_k \ , \tag{37}$$

we can write

$$\mathbf{y}_k = \sum_{\ell=1}^{3} \hat{\mathbf{D}}_{k,\ell}\mathbf{x}_\ell + \hat{\mathbf{z}}_k.$$

The ℓ^{th} transmitter has $N-1$ complex symbols $s_{\ell,1}, \ldots, s_{\ell,N-1}$ to convey over these N matched time slots. For notational convenience, we also define $s_{\ell,N} = 0$. Each symbol has an associated scaling coefficient $\beta_{\ell,n} \in \mathbb{C}$ that will be chosen to induce lattice alignment. Define

$$\mathbf{s}_\ell = \begin{bmatrix} s_{\ell,1} \\ \vdots \\ s_{\ell,N} \end{bmatrix} \qquad \boldsymbol{\beta}_\ell = \begin{bmatrix} \beta_{\ell,1} \\ \vdots \\ \beta_{\ell,N} \end{bmatrix}. \tag{38}$$

To generate the channel input vector, we scale the symbols and then take the DFT,

$$\mathbf{x}_\ell = \mathbf{W}\,\mathrm{diag}(\boldsymbol{\beta}_\ell)\mathbf{s}_\ell.$$

In order to meet the power constraint, each symbol is assumed to have average power P/N.

The k^{th} receiver creates N effective channels by taking the inverse DFT:

$$\tilde{\mathbf{y}}_k = \mathbf{W}^{-1}\mathbf{y}_k \tag{39}$$

$$= \sum_{\ell=1}^{3} \mathbf{W}^{-1}\hat{\mathbf{D}}_{k,\ell}\mathbf{W}\,\mathrm{diag}(\boldsymbol{\beta}_\ell)\mathbf{s}_\ell + \tilde{\mathbf{z}}_k \tag{40}$$

where $\tilde{\mathbf{z}}_k = \mathbf{W}^{-1}\hat{\mathbf{z}}_k$ is a mixture of Gaussian noise and quantization noise. Using the quantization error bound (24), the variance[5] of each element of $\tilde{\mathbf{z}}_k$ is at most

$$\mathrm{Var}(z_{k,n}) \leq \frac{1}{N}\left(1 + \frac{\pi^2 KP}{\nu^2 N^2}\right). \tag{41}$$

From (28), the quantized channel gains satisfy

$$\hat{\mathbf{D}}_{k,\ell} = \begin{cases} \hat{h}_{k,\ell}\mathbf{I} & (k,\ell) \neq (2,3), \\ \hat{h}_{2,3}\mathbf{F} & (k,\ell) = (2,3). \end{cases} \tag{42}$$

Define $\hat{h}_{k,\ell} = \hat{h}_{k,\ell}[t_1]$. For $n = 1, \ldots, N-1$, we can use (33) to express the effective channel outputs as

$$\tilde{y}_{1,n} = \hat{h}_{1,1}\beta_{1,n}s_{1,n} + \hat{h}_{1,2}\beta_{2,n}s_{2,n} + \hat{h}_{1,3}\beta_{3,n}s_{3,n} + \tilde{z}_{1,n}$$

$$\tilde{y}_{2,n} = \hat{h}_{2,1}\beta_{1,n}s_{1,n} + \hat{h}_{2,2}\beta_{2,n}s_{2,n} + \hat{h}_{2,3}\beta_{3,n+1}s_{3,n+1} + \tilde{z}_{2,n}$$

$$\tilde{y}_{3,n} = \hat{h}_{3,1}\beta_{1,n}s_{1,n} + \hat{h}_{3,2}\beta_{2,n}s_{2,n} + \hat{h}_{3,3}\beta_{3,n}s_{3,n} + \tilde{z}_{3,n}.$$

[5] Strictly speaking, the variance is evaluated with respect to both the randomness of the noise and the randomness of the codebooks.

where $\tilde{y}_{k,n}$ and $\tilde{z}_{k,n}$ are the n^{th} entries of $\tilde{\mathbf{y}}_k$ and $\tilde{\mathbf{z}}_k$, respectively.

To determine the scaling and integer coefficients as well as resulting lattice-aligned effective channels, we start by setting $\beta_{3,1} = 1$ and $n = 1$, and then run the following procedure:

1) Align interference in $\tilde{y}_{1,n}$:

If $|\hat{h}_{1,2}| \geq |\hat{h}_{1,3}||\beta_{3,n}|$, set

$$b_{1,n} = \left\lfloor \frac{|\hat{h}_{1,2}|}{|\hat{h}_{1,3}||\beta_{3,n}|} \right\rfloor \qquad \beta_{2,n} = \frac{b_{1,n}\hat{h}_{1,3}\beta_{3,n}}{\hat{h}_{1,2}}$$

$$\tilde{y}_{1,n} = \hat{h}_{1,1}\beta_{1,n}s_{1,n} + \hat{h}_{1,3}\beta_{3,n}(b_{1,n}s_{2,n} + s_{3,n}) + \tilde{z}_{1,n}.$$

Otherwise, if $|\hat{h}_{1,2}| < |\hat{h}_{1,3}||\beta_{3,n}|$, set

$$b_{1,n} = \left\lceil \frac{|\hat{h}_{1,3}||\beta_{3,n}|}{|\hat{h}_{1,2}|} \right\rceil \qquad \beta_{2,n} = \frac{\hat{h}_{1,3}\beta_{3,n}}{b_{1,n}\hat{h}_{1,2}}$$

$$\tilde{y}_{1,n} = \hat{h}_{1,1}\beta_{1,n}s_{1,n} + \hat{h}_{1,2}\beta_{2,n}(s_{2,n} + b_{1,n}s_{3,n}) + \tilde{z}_{1,n}.$$

2) Align interference in $\tilde{y}_{3,n}$:

If $|\hat{h}_{3,1}| \geq |\hat{h}_{3,2}||\beta_{2,n}|$, set

$$b_{3,n} = \left\lfloor \frac{|\hat{h}_{3,1}|}{|\hat{h}_{3,2}||\beta_{2,n}|} \right\rfloor \qquad \beta_{1,n} = \frac{b_{3,n}\hat{h}_{3,2}\beta_{2,n}}{\hat{h}_{3,1}}$$

$$\tilde{y}_{3,n} = \hat{h}_{3,3}\beta_{3,n}s_{3,n} + \hat{h}_{3,2}\beta_{2,n}(b_{3,n}s_{1,n} + s_{2,n}) + \tilde{z}_{3,n}.$$

Otherwise, if $|\hat{h}_{3,1}| < |\hat{h}_{3,2}||\beta_{2,n}|$, set

$$b_{3,n} = \left\lceil \frac{|\hat{h}_{3,2}||\beta_{2,n}|}{|\hat{h}_{3,1}|} \right\rceil \qquad \beta_{1,n} = \frac{\hat{h}_{3,2}\beta_{2,n}}{b_{3,n}\hat{h}_{3,1}}$$

$$\tilde{y}_{3,n} = \hat{h}_{3,3}\beta_{3,n}s_{3,n} + \hat{h}_{3,1}\beta_{1,n}(s_{1,n} + b_{3,n}s_{2,n}) + \tilde{z}_{3,n}.$$

3) Align interference in $\tilde{y}_{2,n}$:

If $|\hat{h}_{2,3}| \geq |\hat{h}_{2,1}||\beta_{1,n}|$, set

$$b_{2,n} = \left\lfloor \frac{|\hat{h}_{2,3}|}{|\hat{h}_{2,1}||\beta_{1,n}|} \right\rfloor \qquad \beta_{3,n+1} = \frac{b_{2,n}\hat{h}_{2,1}\beta_{1,n}}{\hat{h}_{2,3}}$$

$$\tilde{y}_{2,n} = \hat{h}_{2,2}\beta_{2,n}s_{2,n} + \hat{h}_{2,1}\beta_{1,n}(s_{1,n} + b_{2,n}s_{3,n+1}) + \tilde{z}_{2,n}.$$

Otherwise, if $|\hat{h}_{2,3}| < |\hat{h}_{2,1}||\beta_{1,n}|$, set

$$b_{2,n} = \left\lceil \frac{|\hat{h}_{2,1}||\beta_{1,n}|}{|\hat{h}_{2,3}|} \right\rceil \qquad \beta_{3,n+1} = \frac{\hat{h}_{2,1}\beta_{1,n}}{b_{2,n}\hat{h}_{2,3}}$$

$$\tilde{y}_{3,n} = \hat{h}_{3,3}\beta_{3,n}s_{3,n} + \hat{h}_{2,3}\beta_{2,n+1}(b_{2,n}s_{1,n} + s_{3,n+1}) + \tilde{z}_{2,n}.$$

4) If $n < N - 1$, increment n by 1 and loop back to Step 1. Otherwise, terminate the procedure.

For completeness, we also set $\beta_{1,N} = \beta_{2,N} = 1$.

Overall, the k^{th} receiver observes $N-1$ effective channels of the form

$$\tilde{y}_{k,n} = g_{k,n} s_{k,n} + g_{\text{int},k,n} s_{\text{int},k,n} + \tilde{z}_{k,n} \tag{43}$$

where

$$\frac{1}{2}|h_{k,k}| \leq |g_{k,n}| \leq |h_{k,k}| \tag{44}$$

$$\frac{1}{2}\min_{\ell \neq k} |h_{k,\ell}| \leq |g_{\text{int},k,n}| \leq \min_{\ell \neq k} |h_{k,\ell}|, \tag{45}$$

and $s_{\text{int},k,n}$ is an integer-linear combination of (at most) two interfering symbols. From the perspective of each receiver, each symbol appears in exactly one effective channel (if we ignore its small contribution to the quantization error). The phases of $g_{k,n}$ and $g_{\text{int},k,n}$ are independent and uniform over $\mathcal{Q}$. We will ignore the N^{th} channel at each receiver. Next, we will show how to code over these effective channels.

C. Lattice Coding

The users select their codewords from nested lattice codebooks so that integer-linear combinations of codewords are themselves codewords. This allows us to use the compute-and-forward scheme to first decode the integer-linear combination of interfering codewords, cancel out its contribution to the channel observation, and finally decode the desired codeword.

We will code across each quantized channel realization separately. Using the compute-and-forward framework [18], the ℓ^{th} user selects a nested lattice codebook $\mathcal{C}_\ell$ with blocklength $\frac{(1-\gamma)T}{|\hat{\mathcal{H}}|N}$, power P/N, and rate

$$\tilde{R}_\ell = \log(1 + |h_{\ell,\ell}|^2 P) - 3 - \log\left(1 + \frac{\pi^2 K P}{L^2 N^2}\right) . \tag{46}$$

It then splits its message m_ℓ into $(N-1)|\hat{\mathcal{H}}|$ equal-rate submessages $m_{\ell,n}^{(\hat{\mathsf{H}})}$, one for each subchannel $n \in \{1, \ldots, N-1\}$ and quantized channel realization $\hat{\mathsf{H}} \in \hat{\mathcal{H}}$. The submessage $m_{\ell,n}^{(\hat{\mathsf{H}})}$ is then mapped to the corresponding nested lattice codeword $\mathbf{s}_{\ell,n}^{(\hat{\mathsf{H}})} \in \mathcal{C}_\ell$.

For each quantized channel realization $\hat{\mathsf{H}} \in \hat{\mathcal{H}}$, we group together all $\frac{(1-\gamma)T}{|\hat{\mathcal{H}}|N}$ useable time slots from the first subblock for which $\hat{\mathbf{H}}[t] = \hat{\mathsf{H}}$. Using the modulation and alignment scheme described above, the ℓ^{th} transmitter sends the codewords $\mathbf{s}_{\ell,1}^{(\hat{\mathsf{H}})}, \ldots, \mathbf{s}_{\ell,N-1}^{(\hat{\mathsf{H}})}$ over these time slots. It then repeats these codewords over the matched time slots in the remaining $N-1$ subblocks, again following the modulation and alignment scheme. Note that the average transmit power is at most $(N-1)P/N$, which meets the power constraint.

The k^{th} receiver observes its desired codewords over $N-1$ effective channels of the form

$$\tilde{\mathbf{y}}_{k,n}^{(\hat{\mathsf{H}})} = g_{k,n}^{(\hat{\mathsf{H}})} \mathbf{s}_{k,n}^{(\hat{\mathsf{H}})} + g_{\text{int},k,n}^{(\hat{\mathsf{H}})} \mathbf{s}_{\text{int},k,n}^{(\hat{\mathsf{H}})} + \tilde{\mathbf{z}}_{k,n}^{(\hat{\mathsf{H}})} \tag{47}$$

where $|g_{k,n}^{(\hat{\mathsf{H}})}|$ and $|g_{\text{int},k,n}^{(\hat{\mathsf{H}})}|$ satisfy (44) and (45), respectively, $\mathbf{s}_{\text{int},k,n}^{(\hat{\mathsf{H}})}$ is an integer-linear combination of two interfering codewords, and $\tilde{\mathbf{z}}_{k,n}^{(\hat{\mathsf{H}})}$ is the effective noise, which consists of both channel noise and quantization error, and satisfies (41).

Before recovering its desired codeword $\mathbf{s}_{k,n}^{(\hat{\mathsf{H}})}$, the k^{th} receiver first decodes the interference while treating the desired codeword as noise. It follows from [18, Theorem 3] and [58, Lemma 5] that the integer-linear combination $\mathbf{s}_{\text{int},k,n}^{(\hat{\mathsf{H}})}$ can be decoded successfully from $\tilde{\mathbf{y}}_{k,n}^{(\hat{\mathsf{H}})}$ if the rates of all interfering codebooks satisfy

$$\tilde{R}_\ell < \log\left(\mathsf{SINR}_{\text{int},k,n}^{(\hat{\mathsf{H}})}\right) \quad \ell \neq k \tag{48}$$

where

$$\mathsf{SINR}_{\text{int},k,n}^{(\hat{\mathsf{H}})} \triangleq \frac{\left|g_{\text{int},k,n}^{(\hat{\mathsf{H}})}\right|^2 \frac{P}{N}}{\left|g_{k,n}^{(\hat{\mathsf{H}})}\right|^2 \frac{P}{N} + \frac{1}{N}\left(1 + \frac{\pi^2 KP}{L^2N^2}\right)} \tag{49}$$

$$\geq \frac{\min_{\ell\neq k} \frac{1}{4}|h_{k,\ell}|^2 P}{|h_{k,k}|^2 P + \left(1 + \frac{\pi^2 KP}{L^2N^2}\right)} \tag{50}$$

where the inequality follows from (44) and (45). From the very strong interference condition (13), we know that the rate of the ℓ^{th} user satisfies

$$\tilde{R}_\ell < \log\left(1 + \frac{\min_{\ell\neq k} |h_{k,\ell}|^2 P}{|h_{k,k}|^2 P + 1}\right) - 3 - \log\left(1 + \frac{\pi^2 KP}{L^2N^2}\right) \tag{51}$$

$$< \log\left(\frac{\min_{\ell\neq k} |h_{k,\ell}|^2 P}{|h_{k,k}|^2 P + 1}\right) - 2 - \log\left(1 + \frac{\pi^2 KP}{L^2N^2}\right) \tag{52}$$

$$= \log\left(\frac{\min_{\ell\neq k} \frac{1}{4}|h_{k,\ell}|^2 P}{|h_{k,k}|^2 P + 1}\right) - \log\left(1 + \frac{\pi^2 KP}{L^2N^2}\right) \tag{53}$$

$$< \log\left(\mathsf{SINR}_{\text{int},k,n}^{(\hat{\mathsf{H}})}\right) \ , \tag{54}$$

and thus the interference can be decoded with vanishing probability of error.

Next, the k^{th} receiver forms the effective channel $\tilde{\mathbf{y}}_{k,n}^{(\hat{\mathsf{H}})} - g_{\text{int},k,n}^{(\hat{\mathsf{H}})}\mathbf{s}_{\text{int},k,n}^{(\hat{\mathsf{H}})}$ and attempts to decode $\mathbf{s}_{k,n}^{(\hat{\mathsf{H}})}$. Since

$$\tilde{R}_k < \log\left(1 + \frac{\frac{1}{4}|h_{k,k}|^2 P}{1 + \frac{\pi^2 KP}{L^2N^2}}\right) \leq \log\left(1 + \frac{\left|g_{k,n}^{(\hat{\mathsf{H}})}\right|^2 P}{1 + \frac{\pi^2 KP}{L^2N^2}}\right) \ , \tag{55}$$

decoding succeeds with vanishing probability of error.

Summing across all quantized channel realizations and normalizing by the total number of time slots T, we find that the achievable rate for the ℓ^{th} transmitter is

$$R_\ell = \frac{1-\gamma}{|\hat{\mathcal{H}}|N} \sum_{n=1}^{N-1} \sum_{\hat{\mathsf{H}}\in\hat{\mathcal{H}}} \tilde{R}_\ell. \tag{56}$$

Taking the quantization parameter L to zero, we get that

$$\lim_{L\to 0} R_\ell = \frac{1-\gamma}{N} \sum_{n=1}^{N-1} \left(\log(1 + |h_{\ell,\ell}|^2 P) - 3\right) \tag{57}$$

Taking the number of subblocks N to infinity and then the typicality parameter γ to zero, we find that the rates

$$R_\ell = \log(1 + |h_{\ell,\ell}|^2 P) - 3 \tag{58}$$

are achievable for $\ell = 1, \ldots, K$.

VI. Proof of Theorem 1 for $K > 3$ Users

The achievability proof for $K > 3$ users employs the same basic components as in the $K = 3$ user case: channel matching, Fourier modulation and alignment, and lattice coding. The main difference is that, following the alignment scheme of Cadambe and Jafar [3], we need a much larger set of beamforming vectors. We will match up time slots based on quantized channel matrices, using the quantization scheme in Section IV. Recall that this results in two sources of rate loss. First, some time slots are ignored since their quantized channel matrices do not find a match. Second, the alignment and equalization is carried out imperfectly due to quantization error. However, as the typicality parameter γ tends to 0 and the quantization parameter L and the blocklength T tend to infinity, both sources of rate loss vanish. Since the analysis of these rate loss terms can be directly inferred from the $K = 3$ user case, we will assume hereafter that $\hat{h}_{k,\ell}[t] \approx h_{k,\ell}[t]$ (i.e., the quantization parameter L is very large) and that all but a $o(1)$ fraction of time slots are useable (i.e., the typicality parameter γ is very small). This will allows us to streamline the proof.

A. Channel Matching

Let M be a natural number that will be specified later and define

$$N = (M+1)^{K^2} \tag{59}$$

$$d_{k,\ell} = (M+1)^{(\ell-1)K+(k-1)}. \tag{60}$$

We will divide the T channel realizations into N consecutive subblocks, each of length T/N. As before, the n^{th} subblock refers to the time slots from $t = (n-1)T/N + 1$ to $t = nT/N$. Initialize $t_1 = 1$ and search for the first unmatched time slots $t_2, \ldots, t_N$ in subblocks $n = 2, \ldots, N$, respectively, that satisfy

$$h_{k,\ell}[t_n] = \omega^{-(n-1)d_{k,\ell}} h_{k,\ell}[t_1] \quad \forall k, \ell \in \{1, \ldots, K\}. \tag{61}$$

These time slots $t_1, \ldots, t_N$ are set aside as a matched pair, t_1 is incremented by 1, and we repeat the search.[6] This process continues until $t_1 = T/N$ and will match up nearly all time slots (if the sequence of channel realizations is typical).

B. Fourier Modulation and Alignment

Consider any group of matched time slots $t_1, \ldots, t_N$ and define the associated channel input, output, and noise vectors according to (34) as well as the diagonal channel gain matrices according to (35). Due to the matching condition (61), the diagonal channel gain matrices satisfy

$$\mathbf{D}_{k,\ell} = h_{k,\ell} \mathbf{F}^{d_{k,\ell}} \tag{62}$$

where $h_{k,\ell} \triangleq h_{k,\ell}[t_1]$ and $\mathbf{F}$ is defined in (32).

[6]A vanishing fraction of time slots will not be matched, and should be ignored as in Section V-A.

The codeword symbols will be carried along the following set of beamforming vectors:

$$\mathcal{V} = \left\{ \left(\prod_{k,\ell} \beta_{k,\ell,i_{k,\ell}} \mathbf{F}^{d_{k,\ell} i_{k,\ell}} \right) \mathbf{1} : i_{k,\ell} \in \{0, 1, \dots, M-1\} \right\} \tag{63}$$

where the scaling coefficients $\beta_{k,\ell,i_{k,\ell}}$ will be used later on to create lattice-aligned subchannels. It is shown in [53, p.3826] that the M^{K^2} beamforming vectors in $\mathcal{V}$ are orthogonal to one another. Moreover, the values of the indices $\{i_{k,\ell} : k, \ell \in \{1, \dots, K\}\}$ that identify each beamforming vector can be uniquely determined from the sum $\sum_{k,\ell} d_{k,\ell} i_{k,\ell}$.

Each transmitter has M^{K^2} complex symbols $\{s_{\ell,\mathbf{v}} : \mathbf{v} \in \mathcal{V}\}$ to convey over the N matched time slots. As implied by our notation, each beamforming vector is scaled by the associated complex symbol, and the transmitter sends the sum of the resulting vectors,

$$\mathbf{x}_\ell = \sum_{\mathbf{v} \in \mathcal{V}} \mathbf{v}\, s_{\ell,\mathbf{v}}. \tag{64}$$

The channel output vector is thus

$$\mathbf{y}_k = \sum_{\ell=1}^{K} \mathbf{D}_{k,\ell} \mathbf{x}_\ell + \mathbf{z}_k \tag{65}$$

$$= \sum_{\ell=1}^{K} \sum_{\mathbf{v} \in \mathcal{V}} \mathbf{D}_{k,\ell} \mathbf{v}\, s_{\ell,\mathbf{v}} + \mathbf{z}_k \tag{66}$$

$$= \sum_{\ell=1}^{K} \sum_{\mathbf{v} \in \mathcal{V}} h_{k,\ell} \mathbf{F}^{d_{k,\ell}} \mathbf{v}\, s_{\ell,\mathbf{v}} + \mathbf{z}_k. \tag{67}$$

Due to the $\mathbf{F}^{d_{k,\ell}}$ terms, the set of received beamforming vectors

$$\tilde{\mathcal{V}} = \left\{ \left(\prod_{k,\ell} \beta_{k,\ell,i_{k,\ell}} \mathbf{F}^{d_{k,\ell} i_{k,\ell}} \right) \mathbf{1} : i_{k,\ell} \in \{0, 1, \dots, M\} \right\} \tag{68}$$

is larger than the transmitted set $\mathcal{V}$. As before, the $N = (M+1)^{K^2}$ vectors in $\tilde{\mathcal{V}}$ are orthogonal to one another and their indices can be uniquely determined from $\sum_{k,\ell} d_{k,\ell} i_{k,\ell}$.

For each vector $\tilde{\mathbf{v}} \in \tilde{\mathcal{V}}$, the k^{th} receiver forms the effective channel output

$$\tilde{y}_{k,\tilde{\mathbf{v}}} = \frac{1}{\beta_{\tilde{\mathbf{v}}} N} \tilde{\mathbf{v}}^* \mathbf{y}_k \tag{69}$$

$$= \frac{1}{\beta_{\tilde{\mathbf{v}}} N} \sum_{\ell=1}^{K} \sum_{\mathbf{v} \in \mathcal{V}} h_{k,\ell} \tilde{\mathbf{v}}^* \mathbf{F}^{d_{k,\ell}} \mathbf{v}\, s_{\ell,\mathbf{v}} + \tilde{z}_{k,\tilde{\mathbf{v}}} \tag{70}$$

where

$$\beta_{\tilde{\mathbf{v}}} = \prod_{k,\ell} \beta_{k,\ell,\tilde{i}_{k,\ell}} \tag{71}$$

is the product of the scaling coefficients for the indices $\tilde{i}_{k,\ell}$ that identify $\tilde{\mathbf{v}}$ and

$$\tilde{z}_{k,\tilde{\mathbf{v}}} = \frac{1}{\beta_{\tilde{\mathbf{v}}} N} \tilde{\mathbf{v}}^* \mathbf{z}_k. \tag{72}$$

Since $\|\tilde{\mathbf{v}}\| = \beta_{\tilde{\mathbf{v}}} \sqrt{N}$, we have that $\tilde{z}_{k,\tilde{\mathbf{v}}} \sim \mathcal{CN}(0, 1/N)$.

Due to the orthogonality of the beamforming vectors, only one symbol from each transmitter will be observed in this effective channel. For $\ell = 1, \ldots, K$, define

$$\tilde{\mathbf{v}}_\ell = \frac{\beta_{k,\ell,\tilde{i}_{k,\ell}-1}}{\beta_{k,\ell,\tilde{i}_{k,\ell}}} \mathbf{F}^{-d_{k,\ell}} \tilde{\mathbf{v}}, \tag{73}$$

taking $\beta_{k,\ell,-1} = 0$. The effective channel output can be expressed as

$$\tilde{y}_{k,\tilde{\mathbf{v}}} = \frac{1}{\beta_{\tilde{\mathbf{v}}} N} \sum_{\ell=1}^{K} h_{k,\ell} \tilde{\mathbf{v}}^* \mathbf{F}^{d_{k,\ell}} \tilde{\mathbf{v}}_\ell \, s_{\ell,\tilde{\mathbf{v}}_\ell} + \tilde{z}_{k,\tilde{\mathbf{v}}} \tag{74}$$

$$= \frac{\|\tilde{\mathbf{v}}\|^2}{\beta_{\tilde{\mathbf{v}}} N} \sum_{\ell=1}^{K} h_{k,\ell} \frac{\beta_{k,\ell,\tilde{i}_{k,\ell}-1}}{\beta_{k,\ell,\tilde{i}_{k,\ell}}} s_{\ell,\tilde{\mathbf{v}}_\ell} + \tilde{z}_{k,\tilde{\mathbf{v}}} \tag{75}$$

$$= \beta_{\tilde{\mathbf{v}}} \sum_{\ell=1}^{K} h_{k,\ell} \frac{\beta_{k,\ell,\tilde{i}_{k,\ell}-1}}{\beta_{k,\ell,\tilde{i}_{k,\ell}}} s_{\ell,\tilde{\mathbf{v}}_\ell} + \tilde{z}_{k,\tilde{\mathbf{v}}}. \tag{76}$$

Note that each symbol will appear in exactly one effective channel output at each receiver.

We now select the scaling coefficients to induce lattice alignment. First, we set $\beta_{k,k,i_{k,k}} = 1$ for $k = 1, \ldots, K$ and $i_{k,k} = 0, \ldots, M$ to obtain

$$\tilde{y}_{k,\tilde{\mathbf{v}}} = \beta_{\tilde{\mathbf{v}}} \left(h_{k,k} s_{k,\tilde{\mathbf{v}}_k} + \sum_{\ell \neq k} h_{k,\ell} \frac{\beta_{k,\ell,\tilde{i}_{k,\ell}-1}}{\beta_{k,\ell,\tilde{i}_{k,\ell}}} s_{\ell,\tilde{\mathbf{v}}_\ell} \right) + \tilde{z}_{k,\tilde{\mathbf{v}}}.$$

Next, we define

$$\ell_{k,\min} = \arg\min_{\ell \neq k} |h_{k,\ell}| \tag{77}$$

to be the index of the weakest interferer observed at receiver k, and set $\beta_{k,\ell_{k,\min},i_{k,k}} = 1$ for $k = 1, \ldots, K$ and $i_{k,k} = 0, \ldots, M$ as well. Now, for each $k \in \{1, \ldots, K\}$ and $\ell \in \{1, \ldots, K\} \setminus \{k, \ell_{k,\min}\}$, we set

$$\beta_{k,\ell,0} = 1, \tag{78}$$

and recursively select integer and scaling coefficients for $i_{k,\ell} = 1, \ldots, M$ as follows:

$$b_{k,\ell,i_{k,\ell}} = \left\lceil \frac{|h_{k,\ell}|}{|h_{k,\ell_{k,\min}}|} |\beta_{k,\ell,i_{k,\ell}-1}| \right\rceil \tag{79}$$

$$\beta_{k,\ell,i_{k,\ell}} = \frac{h_{k,\ell}\, \beta_{k,\ell,i_{k,\ell}-1}}{h_{k,\ell_{k,\min}}\, b_{k,\ell,i_{k,\ell}}}. \tag{80}$$

This yields the following effective channels:

$$\tilde{y}_{k,\tilde{\mathbf{v}}} = \beta_{\tilde{\mathbf{v}}} \left(h_{k,k} s_{k,\tilde{\mathbf{v}}_k} + h_{k,\ell_{k,\min}} \sum_{\ell \neq k} b_{k,\ell,\tilde{i}_{k,\ell}} s_{\ell,\tilde{\mathbf{v}}_\ell} \right) + \tilde{z}_{k,\tilde{\mathbf{v}}},$$

and guarantees that

$$\frac{1}{2^{K^2}} \leq |\beta_{\tilde{\mathbf{v}}}| \leq 1. \tag{81}$$

Overall, the k^{th} receiver has M^{K^2} desired symbols, each of which is seen across an effective channel associated with a unique beamforming vector from $\tilde{\mathcal{V}}$. By indexing these effective channels and symbols with index $n = 1, \ldots, M^{K^2}$, we obtain effective channels of the form

$$\tilde{y}_{k,n} = g_{k,n} s_{k,n} + g_{\text{int},k,n} s_{\text{int},k,n} + \tilde{z}_{k,n} \tag{82}$$

where

$$\frac{1}{2^{K^2}}|h_{k,k}| \leq |g_{k,n}| \leq |h_{k,k}| \tag{83}$$

$$\frac{1}{2^{K^2}}\min_{\ell\neq k}|h_{k,\ell}| \leq |g_{\text{int},k,n}| \leq \min_{\ell\neq k}|h_{k,\ell}|, \tag{84}$$

and $s_{\text{int},k,n}$ is an integer-linear combination of (at most) $K-1$ interfering symbols. From the perspective of each receiver, each symbol appears in exactly one effective channel. At each receiver, we will ignore the $N - M^{K^2}$ effective channels that do not carry a desired symbol. In the next section, we evaluate the rate attainable over these effective channels via lattice coding.

C. Lattice Coding

The lattice coding scheme for $K > 3$ users is very similar to that for $K = 3$ users. We summarize the key differences below, noting that we have intentionally omitted the effect of channel quantization.

Using the compute-and-forward framework [18], the ℓ^{th} user selects a nested lattice codebook $\mathcal{C}_\ell$ with power P/N and rate

$$\tilde{R}_\ell = \log(1 + |h_{\ell,\ell}|^2 P) - 2K^2 - 1\ . \tag{85}$$

It then splits its message into equal-rate submessages according to the total number of subchannels (which will depend on the channel quantization process). Each submessage is then mapped to a nested lattice codeword.

For a given channel realization H, we group together all useable time slots from the first subblock for which $\mathbf{H}[t] = \mathsf{H}$. Using the modulation and alignment scheme described above, the ℓ^{th} transmitter sends the codewords $\mathbf{s}_{\ell,1}^{(\mathsf{H})}, \dots, \mathbf{s}_{\ell,M^{K^2}}^{(\mathsf{H})}$ over these time slots. It then repeats these codewords over the matched time slots in the remaining $N-1$ subblocks, again following the modulation and alignment scheme. Note that the average transmit power is at most $\frac{M^{K^2}P}{(M+1)^{K^2}}$, which meets the power constraint.

The k^{th} receiver observes its desired codewords over M^{K^2} effective channels of the form

$$\tilde{\mathbf{y}}_{k,n}^{(\mathsf{H})} = g_{k,n}^{(\mathsf{H})}\mathbf{s}_{k,n}^{(\mathsf{H})} + g_{\text{int},k,n}^{(\mathsf{H})}\mathbf{s}_{\text{int},k,n}^{(\mathsf{H})} + \tilde{\mathbf{z}}_{k,n}^{(\mathsf{H})} \tag{86}$$

where $|g_{k,n}^{(\mathsf{H})}|$ and $|g_{\text{int},k,n}^{(\mathsf{H})}|$ satisfy (83) and (84), respectively, $\mathbf{s}_{\text{int},k,n}^{(\mathsf{H})}$ is an integer-linear combination of at most $K-1$ interfering codewords, and $\tilde{\mathbf{z}}_{k,n}^{(\mathsf{H})}$ is the effective noise of variance $1/N$.

As before, the k^{th} receiver first decodes the interference while treating the desired codeword as noise. Using [18, Theorem 3] and [58, Lemma 5], the integer-linear combination $\mathbf{s}_{\text{int},k,n}^{(\mathsf{H})}$ can be decoded with vanishing probability of error from $\tilde{\mathbf{y}}_{k,n}^{(\mathsf{H})}$ if the rates of all interfering codebooks satisfy

$$\tilde{R}_\ell < \log\left(\mathsf{SINR}_{\text{int},k,n}^{(\mathsf{H})}\right) \quad \ell \neq k \tag{87}$$

where

$$\mathsf{SINR}_{\text{int},k,n}^{(\mathsf{H})} \triangleq \frac{\left|g_{\text{int},k,n}^{(\mathsf{H})}\right|^2 \frac{P}{N}}{\left|g_{k,n}^{(\mathsf{H})}\right|^2 \frac{P}{N} + \frac{1}{N})} \tag{88}$$

$$\geq \frac{1}{2^{2K^2}}\frac{\min_{\ell\neq k}|h_{k,\ell}|^2 P}{|h_{k,k}|^2 P + 1} \tag{89}$$

and the inequality follows from (83) and (84). From the very strong interference condition (13), we know that the rate of the ℓ^{th} user satisfies

$$\tilde{R}_\ell < \log\left(1 + \frac{\min_{\ell \neq k} |h_{k,\ell}|^2 P}{|h_{k,k}|^2 P + 1}\right) - 2K^2 - 1 \tag{90}$$

$$< \log\left(\frac{\min_{\ell \neq k} |h_{k,\ell}|^2 P}{|h_{k,k}|^2 P + 1}\right) - 2K^2 \tag{91}$$

$$= \log\left(\mathsf{SINR}_{\text{int},k,n}^{(\mathsf{H})}\right) , \tag{92}$$

which implies the interference is recovered successfully.

The k^{th} receiver then forms the effective channel $\tilde{\mathbf{y}}_{k,n}^{(\mathsf{H})} - g_{\text{int},k,n}^{(\mathsf{H})} \mathbf{s}_{\text{int},k,n}^{(\mathsf{H})}$ and attempts to decode $\mathbf{s}_{k,n}^{(\mathsf{H})}$. Since

$$\tilde{R}_k < \log\left(1 + \frac{1}{2^{2K^2}} |h_{k,k}|^2 P\right) \leq \log\left(1 + \left|g_{k,n}^{(\mathsf{H})}\right|^2 P\right) , \tag{93}$$

decoding succeeds with vanishing probability of error.

Overall, the achievable rate for the ℓ^{th} transmitter is

$$R_\ell = \frac{M^{K^2}}{(M+1)^{K^2}} \tilde{R}_\ell. \tag{94}$$

By sending the parameter M to infinity, it follows that the rates

$$R_\ell = \log(1 + |h_{\ell,\ell}|^2 P) - 2K^2 - 1 \tag{95}$$

are achievable for $\ell = 1, \ldots, K$.

VII. Conclusions

In this paper, we demonstrated that, for regimes where interference alignment is needed, capacity approximation can be more tractable for time-varying, rather than static, channel phases. We approximated the capacity region of the K-user very strong Gaussian interference channel with time-varying phase to within $2K^2 - 1$ bits per user. It seems likely that this approach can be applied to approximate the capacity in other interference regimes. For instance, if all receivers are in the strong interference regime, we conjecture that the approximate capacity region takes the form of the two-user strong regime capacity region, applied pairwise to all users.

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References

[1] R. H. Etkin, D. N. C. Tse, and H. Wang, "Gaussian interference channel capacity to within one bit," *IEEE Transactions on Information Theory*, vol. 54, pp. 5534–5562, December 2008.

[2] M. A. Maddah-Ali, A. S. Motahari, and A. K. Khandani, "Communication over MIMO X channels: Interference alignment, decomposition, and performance analysis," *IEEE Transactions on Information Theory*, vol. 54, pp. 3457–3470, August 2008.

[3] V. R. Cadambe and S. A. Jafar, "Interference alignment and the degrees of freedom for the K-user interference channel," *IEEE Transactions on Information Theory*, vol. 54, pp. 3425–3441, August 2008.

[4] R. Etkin and E. Ordentlich, "The degrees-of-freedom of the K-user Gaussian interference channel is discontinuous at rational channel coefficients," *IEEE Transactions on Information Theory*, vol. 55, pp. 4932–4946, November 2009.

[5] Y. Wu, S. Shamai (Shitz), and S. Verdú, "Information dimension and the degrees of freedom of the interference channel," *IEEE Transactions on Information Theory*, vol. 61, pp. 256–279, January 2015.

[6] D. Stotz and H. Bölcskei, "Characterizing degrees of freedom through additive combinatorics," *arXiv preprint arXiv:1506.01866*, 2015.

[7] A. J. Goldsmith and P. P. Varaiya, "Capacity of fading channels with channel side information," *IEEE Transactions on Information Theory*, vol. 43, pp. 1986 – 1992, June 1997.

[8] E. Biglieri, J. Proakis, and S. Shamai, "Fading channels: Information-theoretic and communications aspects," *IEEE Transactions on Information Theory*, vol. 44, pp. 2619–2692, October 1998.

[9] R. Knopp and P. Humblet, "Information capacity and power control in single-cell multiuser communications," in *Proceedings of the IEEE International Conference on Communications (ICC 1995)*, (Seattle, WA), pp. 331–335, 1995.

[10] D. Tse and S. Hanly, "Multiaccess fading channels. i. polymatroid structure, optimal resource allocation and throughput capacities," *IEEE Transactions on Information Theory*, vol. 44, pp. 2796 – 2815, July 1998.

[11] A. J. Goldsmith, "The capacity of downlink fading channels with variable rate and power," *IEEE Transactions on Vehicular Technology*, vol. 46, pp. 569–580, August 1997.

[12] D. N. C. Tse, "Optimal power allocation over parallel Gaussian broadcast channels," tech. rep., University of California - Berkeley, 1999.

[13] L. Li and A. J. Goldsmith, "Capacity and optimal resource allocation for fading broadcast channels. i. ergodic capacity," *IEEE Transactions on Information Theory*, vol. 47, pp. 1083–1102, March 2001.

[14] L. Sankar, X. Shang, E. Erkip, and H. V. Poor, "Ergodic two-user interference channels: Is separability optimal?," in *46th Annual Allerton Conference on Communications, Control, and Computing*, (Monticello, IL), September 2008.

[15] V. R. Cadambe and S. A. Jafar, "Parallel Gaussian interference channels are not always separable," *IEEE Transactions on Information Theory*, vol. 55, pp. 3983–3990, September 2009.

[16] V. R. Cadambe and S. A. Jafar, "Sum-capacity and the unique separability of the parallel Gaussian MAC-Z-BC network," in *Proceedings of the IEEE International Symposium on Information Theory (ISIT 2010)*, (Austin, TX), June 2010.

[17] B. Nazer, M. Gastpar, S. A. Jafar, and S. Vishwanath, "Ergodic interference alignment," *IEEE Transactions on Information Theory*, vol. 58, pp. 6355–6371, October 2012.

[18] B. Nazer and M. Gastpar, "Compute-and-forward: Harnessing interference through structured codes," *IEEE Transactions on Information Theory*, vol. 57, pp. 6463–6486, October 2011.

[19] S. A. Jafar, "The ergodic capacity of phase-fading interference networks," *IEEE Transactions on Information Theory*, vol. 57, pp. 7685–7694, December 2011.

[20] A. B. Carleial, "A case where interference does not reduce capacity," *IEEE Transactions on Information Theory*, vol. 21, pp. 569–570, September 1975.

[21] R. Ahlswede, "The capacity region of a channel with two senders and two receivers," *Annals of Probability*, vol. 2, pp. 805–814, October 1974.

[22] H. Sato, "Two-user communication channels," *IEEE Transactions on Information Theory*, vol. 23, pp. 295–304, May 1977.

[23] A. B. Carleial, "Interference channels," *IEEE Transactions on Information Theory*, vol. 24, pp. 60–70, January 1978.

[24] H. Sato, "The capacity of the Gaussian interference channel under strong interference," *IEEE Transactions on Information Theory*, vol. 27, pp. 786–788, November 1981.

[25] T. S. Han and K. Kobayashi, "A new achievable rate region for the interference channel," *IEEE Transactions on Information Theory*, vol. 27, pp. 49–60, January 1981.

[26] M. Costa and A. El Gamal, "The capacity region of the discrete memoryless interference channel with strong interference," *IEEE Transactions on Information Theory*, vol. 33, pp. 710–711, September 1987.

[27] M. Costa and A. El Gamal, "The capacity region of a class of deterministic interference channels," *IEEE Transactions on Information Theory*, vol. 28, pp. 343–346, March 1982.

[28] A. S. Motahari and A. K. Khandani, "Capacity bounds for the Gaussian interference channel," *IEEE Transactions on Information Theory*, vol. 55, pp. 620–643, February 2009.

[29] X. Shang, G. Kramer, and B. Chen, "A new outer bound and the noisy-interference sum-rate capacity for Gaussian interference channels," *IEEE Transactions on Information Theory*, vol. 55, pp. 689–699, February 2009.

[30] V. S. Annapureddy and V. V. Veeravalli, "Gaussian interference networks: Sum capacity in the low-interference regime and new outer bounds on the capacity region," *IEEE Transactions on Information Theory*, vol. 55, pp. 3032–3050, July 2009.

[31] A. Raja, V. M. Prabhakaran, and P. Viswanath, "The two-user compound interference channel," *IEEE Transactions on Information Theory*, vol. 55, pp. 5100–5120, November 2009.

[32] C. Suh and D. N. C. Tse, "Feedback capacity of the Gaussian interference channel to within 2 bits," *IEEE Transactions on Information Theory*, vol. 57, pp. 2667–2685, May 2011.

[33] V. M. Prabhakaran and P. Viswanath, "Interference channels with source cooperation," *IEEE Transactions on Information Theory*, vol. 57, pp. 156–186, January 2011.

[34] I.-H. Wang and D. N. C. Tse, "Interference mitigation through limited transmitter cooperation," *IEEE Transactions on Information Theory*, vol. 57, pp. 2941–2965, May 2011.

[35] I.-H. Wang and D. N. C. Tse, "Interference mitigation through limited receiver cooperation," *IEEE Transactions on Information Theory*, vol. 57, pp. 2913–2940, May 2011.

[36] S. Karmakar and M. K. Varanasi, "The capacity region of the MIMO interference channel and its reciprocity to within a constant gap," *IEEE Transactions on Information Theory*, vol. 59, pp. 4781–4797, August 2013.

[37] Y. Polyanskiy and Y. Wu, "Wasserstein continuity of entropy and outer bounds for interfrence channels," *IEEE Transactions on Information Theory*, vol. 62, pp. 3992–4002, July 2016.

[38] I. Sason, "On achievable rate regions for the Gaussian interference channel," *IEEE Transactions on Information Theory*, vol. 50, pp. 1345–1356, June 2004.

[39] M. H. Costa, "On the Gaussian interference channel," *IEEE Transactions on Information Theory*, vol. 31, pp. 607–615, September 1985.

[40] V. R. Cadambe and S. A. Jafar, "Interference alignment and the degrees of freedom of wireless X networks," *IEEE Transactions on Information Theory*, vol. 55, pp. 2334–2344, May 2009.

[41] T. Gou and S. Jafar, "Degrees of freedom of the K-user MIMO interference channel," *IEEE Transactions on Information Theory*, vol. 56, pp. 6040–6057, December 2010.

[42] T. Gou, S. A. Jafar, S.-W. Jeon, and S.-Y. Chung, "Aligned interference neutralization and the degrees of freedom of the 2x2x2 interference channel," *IEEE Transactions on Information Theory*, vol. 58, pp. 4381–4395, July 2012.

[43] L. Ke, A. Ramamoorthy, Z. Wang, and H. Yin, "Degrees of freedom region for an interference network with general message demands," *IEEE Transactions on Information Theory*, vol. 58, pp. 3787–3797, June 2012.

[44] I. Shomorony and S. Avestimehr, "Degrees of freedom of two-hop wireless networks: Everyone gets the entire cake," *IEEE Transactions on Information Theory*, vol. 60, pp. 2417–2431, May 2014.

[45] S. A. Jafar, "Interference alignment - a new look at signal dimensions in a communication network," in *Foundations and Trends in Communications and Information Theory*, vol. 7, NOW Publishers, 2011.

[46] G. Bresler, A. Parekh, and D. Tse, "The approximate capacity of the many-to-one and one-to-many Gaussian interference channels," *IEEE Transactions on Information Theory*, vol. 56, pp. 4566–4592, September 2010.

[47] A. S. Motahari, S. Oveis-Gharan, M.-A. Maddah-Ali, and A. K. Khandani, "Real interference alignment: Exploiting the potential of single antenna systems," *IEEE Transactions on Information Theory*, vol. 60, pp. 4799–4810, August 2014.

[48] U. Niesen and M. A. Maddah-Ali, "Interference alignment: From degrees-of-freedom to constant-gap capacity approximations," *IEEE Transactions on Information Theory*, vol. 59, pp. 4855–4888, August 2013.

[49] O. Ordentlich, U. Erez, and B. Nazer, "The approximate sum capacity of the symmetric K-user Gaussian interference channel," *IEEE Transactions on Information Theory*, vol. 60, pp. 3450–3482, June 2014.

[50] C. Geng, N. Naderializadeh, A. S. Avestimehr, and S. A. Jafar, "On the optimality of treating interference as noise," *IEEE Transactions on Information Theory*, vol. 61, pp. 1753–1767, April 2015.

[51] O. Johnson, M. Aldridge, and R. Piechocki, "Interference alignment-based sum capacity bounds for random dense gaussian interference networks," *IEEE Transactions on Information Theory*, vol. 57, pp. 282–290, January 2011.

[52] U. Niesen, "Interference alignment in dense wireless networks," *IEEE Transactions on Information Theory*, vol. 57, pp. 2889–2901, May 2011.

[53] U. Niesen, B. Nazer, and P. Whiting, "Computation alignment: Capacity approximation without noise accumulation," *IEEE Transactions on Information Theory*, vol. 59, pp. 3811–3832, 2013 2013.

[54] R. Bassily and S. Ulukus, "Ergodic secret alignment," *IEEE Transactions on Information Theory*, vol. 58, pp. 1594–1611, March 2012.

[55] S. Sridharan, A. Jafarian, S. Vishwanath, and S. A. Jafar, "Capacity of symmetric K-user Gaussian very strong interference channels," in *Proceedings of the IEEE Global Communications Conference (GLOBECOM 2008)*, (New Orleans, LA), December 2008.

[56] A. El Gamal and Y.-H. Kim, *Network Information Theory*. Cambridge University Press, 2011.

[57] M. Vetterli, J. Kovačević, and V. K. Goyal, *Foundations of Signal Processing*. Cambridge, UK: Cambridge University Press, 2014.

[58] V. Ntranos, V. Cadambe, B. Nazer, and G. Caire, "Expanding the compute-and-forward framework: Unequal powers, signal levels, and multiple linear combinations," *IEEE Transactions on Information Theory*, to appear 2016. Available online: http://arxiv.org/abs/1504.01690.