Integer-Forcing Linear Receivers

Jiening Zhan, Bobak Nazer, Uri Erez, Michael Gastpar

ISIT 2010
Austin, Texas

June 14, 2010
MIMO Receiver Architectures
MIMO Receiver Architectures

- Single stream encoding: each antenna encodes an independent data stream
MIMO Receiver Architectures

- Single stream encoding: each antenna encodes an independent data stream
- Symmetric rate and equal power situation
MIMO Receiver Architectures

- Single stream encoding: each antenna encodes an independent data stream
- Symmetric rate and equal power situation
- Decoder recovers the messages reliably:

\[ Pr \left( (\hat{w}_1, ..., \hat{w}_M) \neq (w_1, ..., w_M) \right) \leq \epsilon \quad \text{for all } \epsilon > 0 \]
MIMO Receiver Architectures

- Single stream encoding: each antenna encodes an independent data stream
- Symmetric rate and equal power situation
- Decoder recovers the messages reliably:
  \[ Pr \left( \left( \hat{w}_1, ..., \hat{w}_M \right) \neq \left( w_1, ..., w_M \right) \right) \leq \epsilon \quad \text{for all } \epsilon > 0 \]
- Focus on receiver architecture design
MIMO Receiver Architectures

Joint detection and decoding

Separate detection and decoding

Detection

Decoding

message 1

message M

message 1

message M
MIMO Receiver Architectures

Joint detection and decoding

Separate detection and decoding

Complexity

Performance

Separate (Linear)

Joint

Detection and Decoding

message 1

message M

Detection

Decoding

message 1

message M

UC Berkeley

Zhan, Nazer, Erez, Gastpar
MIMO Receiver Architectures

Joint detection and decoding

Separate detection and decoding

Complexity

Performance

Separate (Linear)

Joint
MIMO Receiver Architectures

Joint detection and decoding

Separate detection and decoding

Complexity

Performance

Separate (Linear)

Joint

Proposed

UC Berkeley

Zhan, Nazer, Erez, Gastpar
Joint ML Receiver

\[
\begin{bmatrix}
\cdots & y_1(w_1) & \cdots & \cdots \\
\cdots & y_2(w_2) & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
\cdots & y_N(w_N) & \cdots & \cdots \\
\end{bmatrix}
= \begin{bmatrix}
\cdots & x_1(w_1) & \cdots & \cdots \\
\cdots & x_2(w_2) & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
\cdots & x_M(w_M) & \cdots & \cdots \\
\end{bmatrix}
+ Z
\]

- Performs a search for the most likely set of transmitted messages \(w_1, \ldots, w_M\)
Joint ML Receiver

\[
\begin{bmatrix}
\cdots & y_1(w_1) & \cdots & \cdots \\
\cdots & y_2(w_2) & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
\cdots & y_N(w_N) & \cdots & \cdots \\
\end{bmatrix}
= H
\begin{bmatrix}
\cdots & x_1(w_1) & \cdots & \cdots \\
\cdots & x_2(w_2) & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
\cdots & x_M(w_M) & \cdots & \cdots \\
\end{bmatrix}
+ Z
\]

- Performs a search for the most likely set of transmitted messages \( w_1, \ldots, w_M \)
- Optimal in terms of capacity and probability of error
Joint ML Receiver

\[
\begin{bmatrix}
\cdots & y_1(w_1) & \cdots & \cdots \\
\cdots & y_2(w_2) & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
\cdots & y_N(w_N) & \cdots & \cdots
\end{bmatrix} = H
\begin{bmatrix}
\cdots & x_1(w_1) & \cdots & \cdots \\
\cdots & x_2(w_2) & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
\cdots & x_M(w_M) & \cdots & \cdots
\end{bmatrix} + Z
\]

- Performs a search for the most likely set of transmitted messages \(w_1, \ldots, w_M\)
- Optimal in terms of capacity and probability of error
- Merges the space and time dimensions in processing
Joint ML Receiver

\[
\begin{bmatrix}
\cdots & y_1(w_1) & \cdots & \cdots \\
\cdots & y_2(w_2) & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
\cdots & y_N(w_N) & \cdots & \cdots \\
\end{bmatrix} = H \begin{bmatrix}
\cdots & x_1(w_1) & \cdots & \cdots \\
\cdots & x_2(w_2) & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
\cdots & x_M(w_M) & \cdots & \cdots \\
\end{bmatrix} + Z
\]

- Performs a search for the most likely set of transmitted messages \(w_1, \ldots, w_M\)
- Optimal in terms of capacity and probability of error
- Merges the space and time dimensions in processing
- Complexity is exponential in the \textit{product} of the number of data streams \(M\) and the blocklength \(n\)
Linear Receivers

- Original Channel:

\[ y = Hx + z \]
Linear Receivers

- Original Channel:
  \[ y = Hx + z \]

- Linear Projection:
  \[ \tilde{y} = BHx + Bz \]
Linear Receivers

- Original Channel:
  \[ y = Hx + z \]

- Linear Projection:
  \[ \tilde{y} = BHx + Bz \]

- Each element \( \tilde{y}_1, \ldots, \tilde{y}_M \) is then sent to a separate decoder.
Linear Receivers

- Original Channel:
  \[ y = Hx + z \]

- Linear Projection:
  \[ \tilde{y} = BHx + Bz \]

- Each element \( \tilde{y}_1, \ldots, \tilde{y}_M \) is then sent to a separate decoder

- Separates space and time in processing
Linear Receivers

- Original Channel:
  \[ y = Hx + z \]

- Linear Projection:
  \[ \tilde{y} = BHx + Bz \]

- Each element \( \tilde{y}_1, \ldots, \tilde{y}_M \) is then sent to a separate decoder
- Separates space and time in processing
- Reduces to \( M \) single input single output channels
Linear Receivers

- **Original Channel:**
  \[ y = Hx + z \]

- **Linear Projection:**
  \[ \tilde{y} = BHx + Bz \]

- Each element \( \tilde{y}_1, \ldots, \tilde{y}_M \) is then sent to a separate decoder
- Separates space and time in processing
- Reduces to \( M \) single input single output channels
- Complexity is at most exponential in the blocklength \( n \)
Linear Receivers

- Original Channel:
  \[ y = Hx + z \]

- Linear Projection:
  \[ \tilde{y} = BHx + Bz \]

- Each element \( \tilde{y}_1, \ldots, \tilde{y}_M \) is then sent to a separate decoder

- Separates space and time in processing

- Reduces to \( M \) single input single output channels

- Complexity is at most exponential in the blocklength \( n \)

- Decorrelator sets \( B = H^{-1} \), MMSE sets \( B = H^* \left( HH^* + \frac{1}{\text{SNR}} I \right)^{-1} \).
Linear Receivers
Linear Receivers

- Traditional LR results in large SNR loss in the received data streams when the channel vectors are correlated
- Integer LR mitigates this SNR loss
Linear Receivers

- Traditional LR results in large SNR loss in the received data streams when the channel vectors are correlated.
- Integer LR mitigates this SNR loss.
- Decode linear equations of the transmitted data rather than the data itself.
Example

\[ \mathbf{H} \]

\[ \begin{align*}
\mathbf{y} &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \mathbf{z} \\
\text{Traditional LR} \\
\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \mathbf{y} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \mathbf{z} \\
\sigma_1^2 &= 5 \\
\sigma_2^2 &= 2 \\
\text{Integer LR} \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 + 2x_2 \\ x_1 + x_2 \end{bmatrix} + \mathbf{z} \\
\sigma_1^2 &= 1 \\
\sigma_2^2 &= 1
\end{align*} \]
Compute-and-Forward

[Nazer-Gastpar 08]

- Each transmitter uses the same linear code.
- Receiver observes a noisy linear combination of transmitted codewords:
  \[ Y^n = \sum_{m=1}^{M} h_m X^n_m + Z^n \]
- Reliably decode a linear function of the codewords:
  \[ U^n = \sum_{m=1}^{M} a_m X^n_m \quad \text{where} \quad a_m \in \mathbb{Z} + j\mathbb{Z} \]
Compute-and-Forward

[Nazer-Gastpar 08]

- Each transmitter uses the same linear code.
- Receiver observes a noisy linear combination of transmitted codewords:

\[ Y^n = \sum_{m=1}^{M} h_m X_m^n + Z^n \]

- Reliably decode a linear function of the codewords:

\[ U^n = \sum_{m=1}^{M} a_m X_m^n \quad \text{where} \quad a_m \in \mathbb{Z} + j\mathbb{Z} \]
Compute-and-Forward

[Nazer-Gastpar 08]

Each transmitter uses the same linear code

Receiver observe a noisy linear combination of transmitted codewords:

\[ Y^n = \sum_{m=1}^{M} h_m X_m^n + Z^n \]

Reliably decode a linear function of the codewords

\[ U^n = \sum_{m=1}^{M} a_m X_m^n \text{ where } a_m \in \mathbb{Z} + j\mathbb{Z} \]

Achievable Rate:

\[ R(h, a) = \max_b \log \left( \frac{\text{SNR}}{b^2 + \text{SNR} \|bh - a\|^2} \right) \]
Compute-and-Forward

[Nazer-Gastpar 08]

- Each transmitter uses the same linear code
- Receiver observe a noisy linear combination of transmitted codewords:
  \[ Y^n = \sum_{m=1}^{M} h_m X_m^n + Z^n \]
- Reliably decode a linear function of the codewords
  \[ U^n = \sum_{m=1}^{M} a_m X_m^n \quad \text{where} \quad a_m \in \mathbb{Z} + j\mathbb{Z} \]

- Achievable Rate:
  \[ R(h, a) = \max_b \log \left( \frac{\text{SNR}}{b^2 + \text{SNR} \|bh - a\|^2} \right) \]
- Uses Nested Lattice Codes [Erez-Zamir 04, Erez - Litsyn- Zamir 05]
Random Codes vs. Structured Codes
Random Codes vs. Structured Codes
Random Codes vs. Structured Codes
Random Codes vs. Structured Codes

- Sum of codewords is not a codeword.
- Must decode individual messages.
Random Codes vs. Structured Codes

- Sum of codewords is not a codeword.
- Must decode individual messages.
Random Codes vs. Structured Codes

- Sum of codewords is not a codeword.
- Must decode individual messages.
Random Codes vs. Structured Codes

- Sum of codewords is not a codeword.
- Must decode individual messages.

- Sum of codewords is a codeword.
- Can decode integer combinations of messages.
Random Codes vs. Structured Codes

- Sum of codewords is not a codeword.
- Must decode individual messages.

- Sum of codewords is a codeword.
- Can decode integer combinations of messages.
Each decoder recovers an equation of the transmitted messages
Proposed Architecture

- Each decoder recovers an equation of the transmitted messages
- Freedom to choose \( A \) to be any full rank integer matrix
Proposed Architecture

- Each decoder recovers an equation of the transmitted messages
- Freedom to choose $A$ to be any full rank integer matrix
- Reduces to the Decorrelator when $A = I$ and $B = H^{-1}$
- Reduces to the MMSE estimator when $A = I$ and $B = H^* \left( HH^* + \frac{1}{SNR} I \right)^{-1}$. 
Achievable Rates

- To recover equation with coefficients $a_m$, need that $R < R(H, a_m)$ where

$$R(H, a_m) = \max_{b_m} \log \left( \frac{\text{SNR}}{\|b_m\|^2 + \text{SNR}\|H^Tb_m - a_m\|^2} \right)$$
Achievable Rates

• To recover equation with coefficients $a_m$, need that $R < R(H, a_m)$ where

$$R(H, a_m) = \max_{b_m} \log \left( \frac{\text{SNR}}{\|b_m\|^2 + \text{SNR}\|H^Tb_m - a_m\|^2} \right)$$

• Need a full rank set $A = [a_1, ..., a_M]$. The rate must satisfy the minimum over all constraints:

$$R(H, A) \triangleq \min_m R(H, a_m)$$

$$R < R(H, A)$$
Achievable Rates

- To recover equation with coefficients $a_m$, need that $R < R(H, a_m)$ where

$$R(H, a_m) = \max_{b_m} \log \left( \frac{\text{SNR}}{\|b_m\|^2 + \text{SNR} \|H^T b_m - a_m\|^2} \right)$$

- Need a full rank set $A = [a_1, ..., a_M]$. The rate must satisfy the minimum over all constraints:

$$R(H, A) \triangleq \min_m R(H, a_m)$$

$$R < R(H, A)$$

- Select the best full rank integer matrix:

$$R = \max_{A: |A| > 0} R(H, A)$$

- Only need to search $\|a_m\|^2 < 1 + (\max_i \lambda_i) \text{SNR}$
Integer-Forcing Linear Receivers

Noise variance in received stream $m$:

$$\tilde{\sigma}_m^2 = |v_{\text{min}}^T a_m|^2 \frac{1}{\lambda_{\text{min}}} + |v_{\text{max}}^T a_m|^2 \frac{1}{\lambda_{\text{max}}}$$
Integer-Forcing Linear Receivers

Noise variance in received stream \( m \):

\[
\tilde{\sigma}_m^2 = \left| \mathbf{v}_{\text{min}}^T \mathbf{a}_m \right|^2 \frac{1}{\lambda_{\text{min}}} + \left| \mathbf{v}_{\text{max}}^T \mathbf{a}_m \right|^2 \frac{1}{\lambda_{\text{max}}}
\]
Integer-Forcing Linear Receivers

Noise variance in received stream \( m \):

\[
\tilde{\sigma}_m^2 = |v_{\text{min}}^T a_m|^2 \frac{1}{\lambda_{\text{min}}} + |v_{\text{max}}^T a_m|^2 \frac{1}{\lambda_{\text{max}}}
\]

Decorrelator

[Diagram of a decorrelator matrix with eigenvalues and eigenvectors labeled]
Integer-Forcing Linear Receivers

Noise variance in received stream $m$:

$$\tilde{\sigma}_m^2 = |v_{\text{min}}^T a_m|^2 \frac{1}{\lambda_{\text{min}}} + |v_{\text{max}}^T a_m|^2 \frac{1}{\lambda_{\text{max}}}$$

Decorrelator

Integer

\[ \frac{1}{\lambda_{\text{min}}} v_{\text{min}} \]

\[ \frac{1}{\lambda_{\text{max}}} v_{\text{max}} \]

\[ a_1 \quad a_2 \]

\[ \frac{1}{\lambda_{\text{max}}} v_{\text{max}} \]
Quasi-Static Fading

- Consider the standard i.i.d Rayleigh Fading Model where $h_{i,j} \sim \mathcal{CN}(0, 1)$
- Assume channel is fixed during transmission block
- Only receiver knows the channel matrix
Quasi-Static Fading

- Consider the standard i.i.d Rayleigh Fading Model where $h_{i,j} \sim \mathcal{CN}(0, 1)$
- Assume channel is fixed during transmission block
- Only receiver knows the channel matrix
- The outage probability is given by:

$$
\rho_{\text{OUT}}(R) = \Pr(R(H) < R).
$$
Quasi-Static Fading

- Consider the standard i.i.d Rayleigh Fading Model where $h_{i,j} \sim \mathcal{CN}(0, 1)$
- Assume channel is fixed during transmission block
- Only receiver knows the channel matrix
- The outage probability is given by:
  \[ \rho_{\text{OUT}}(R) = \Pr(R(H) < R). \]
- Define the $\rho$-outage rate to be:
  \[ R(\rho_{\text{OUT}}) = \sup\{ R : \rho_{\text{OUT}} \leq \rho \}. \]
Outage Rates

- Joint
- Integer
- MMSE
- Decorrelator

UC Berkeley
Zhan, Nazer, Erez, Gastpar
Outage Probability
## Diversity Multiplexing Tradeoff

**Theorem**

Consider a MIMO channel with $M$ transmit and $N \geq M$ receive antennas, the achievable diversity multiplexing tradeoff for the integer-forcing receiver is given by

$$d_{\text{INTEGER}}(r) = N \left(1 - \frac{r}{M}\right)$$

where $r \in [0, M]$
Diversity Multiplexing Tradeoff

Theorem

Consider a MIMO channel with $M$ transmit and $N \geq M$ receive antennas, the achievable diversity multiplexing tradeoff for the integer-forcing receiver is given by

$$d_{\text{INTEGER}}(r) = N \left(1 - \frac{r}{M}\right)$$

where $r \in [0, M]$
Diversity Multiplexing Tradeoff

**Theorem**

Consider a MIMO channel with \( M \) transmit and \( N \geq M \) receive antennas, the achievable diversity multiplexing tradeoff for the integer-forcing receiver is given by

\[
d_{\text{INTEGER}}(r) = N \left(1 - \frac{r}{M}\right)
\]

where \( r \in [0, M] \)

\[
d_{\text{JOINT}}(r) = N \left(1 - \frac{r}{M}\right)
\]

\[
d_{\text{DECORR}}(r) = 1 - \frac{r}{M}
\]
Diversity Multiplexing Tradeoff

\[
\begin{align*}
\text{Multiplexing rate } r & \\
\text{Diversity } d & 
\end{align*}
\]

- Integer
- Decorrelator
- Joint

UC Berkeley 
Zhan, Nazer, Erez, Gastpar
Complexity

- Calculating projection matrix $\mathbf{B}$ - same as that of traditional linear architectures.
- Calculating $\mathbf{A}^{-1}$ - low complexity since $\mathbf{A}$ is an integer matrix
- Searching for $\mathbf{A}$ - independent of blocklength $n$. 

\[ \hat{w}_1 \]
\[ \hat{w}_M \]
Future Work

• Develop a low-complexity heuristic search algorithm for matrix $A$
  • Rounding the channel matrix
  • Rounding the maximum eigenvector
• Understand the performance of our receiver under practical codes and constellations.

• Study the case where we allow for coding across transmit antennas. Develop efficient space-time codes to be used in conjunction with lattice codes.

• Study the effect of distributed decoders.