The Impact of Channel Variation on Integer-Forcing Receivers

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Abstract—Consider several single-antenna transmitters that wish to simultaneously communicate with a multiple-antenna receiver. Recent work has proposed the integer-forcing linear receiver architecture as an alternative to conventional linear receivers. The key idea is to first use linear equalization to obtain an integer-valued effective channel, then employ single-user decoders to recover integer-linear combinations of the messages, and finally solve these for the desired messages. For the special case where the channel matrix remains fixed for the duration of the codeword, it has been shown that integer-forcing can operate very close to the performance of optimal joint maximum likelihood decoding. In this paper, we investigate the impact of channel variation on the integer-forcing linear receiver and show it still retains an advantage over conventional linear receivers, despite the fact that the integer coefficients must remain fixed across the codeword duration.

I. INTRODUCTION

Linear receivers are often employed as a means of reducing the implementation complexity of multiple-input multiple-output (MIMO) decoding. The basic idea is to first separate the data streams via linear equalization and then recover them via single-user decoding. However, in many scenarios, conventional linear receivers fall short of the performance of optimal joint maximum likelihood decoding. In this paper, we investigate the impact of channel variation on the integer-forcing linear receiver and show it still retains an advantage over conventional linear receivers.

II. PROBLEM STATEMENT

We will denote column vectors by boldface lowercase (e.g., $x$) and matrices by boldface uppercase (e.g., $X$). Let $X^T$ denote the transpose of a matrix $X$ and let $X_1^N \triangleq (X_1, \ldots, X_N)$ denote a sequence of matrices. For simplicity, we focus on real-valued channels and note that complex-valued channels can be handled via their real-valued decomposition [1].

There are $M_{\text{Rx}}$ single-antenna transmitters that communicate to a receiver equipped with $M_{\text{Rx}}$ antennas. The $t^{th}$ transmitter has a message $w_t \in \mathbb{Z}_p^k$ where $p$ is prime. Using an encoder $E_t : \mathbb{Z}_p^k \rightarrow \mathbb{R}^T$, it maps its message into a sequence of channel inputs, $(x_t[1], \ldots, x_t[T]) = E_t(w_t)$ where $T$ is the coding blocklength. Each transmitter’s input must satisfy the usual power constraint, $\frac{1}{T} \sum_{t=1}^{T} (x_t[t])^2 \leq \text{SNR}$. The rate of each message is $R = \frac{1}{T} \log p$.

At time $t$, the receiver observes

$$y[t] = H[t]x[t] + z[t]$$

where $H[t] \in \mathbb{R}^{M_{\text{Rx}} \times M_{\text{Rx}}}$ is the channel matrix, $x[t] \triangleq [x_1[t] \ldots x_{M_{\text{Rx}}}[t]]^T$ is the vector of channel inputs, and $z[t] \in \mathbb{R}^{M_{\text{Rx}}}$ is independent and identically distributed (i.i.d.) Gaussian noise, $z[t] \sim \mathcal{N}(0, I)$.

We consider two special cases of channel matrix fading distributions: static and block fading. In both cases, we assume that only CSIR is available (although the transmitters are aware of the fading statistics).

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1In practice, it may be possible to implement this decoder via spatially-coupled low-density parity-check (LDPC) coding [7].
The receiver uses a decoder $\hat{w}_n$ to separate the data streams and obtain the effective channel realization over the duration of the codeword. Specifically, for $t = (n - 1)[T/N] + 1, \ldots, n[T/N]$, the channel matrix is $H[t] = H_n$. 

For notational convenience, we define

$X_n \triangleq \begin{bmatrix} x[(n-1)T/N+1] & \cdots & x[nT/N] \end{bmatrix}$

$Y_n \triangleq \begin{bmatrix} y[(n-1)T/N+1] & \cdots & y[nT/N] \end{bmatrix}$

$Z_n \triangleq \begin{bmatrix} z[(n-1)T/N+1] & \cdots & z[nT/N] \end{bmatrix}$. 

This permits us to express the input-output relationship during the $n^{th}$ subblock as

$Y_n = H_n X_n + Z_n$. 

The receiver uses a decoder $D : \mathbb{R}^{M_k \times T} \rightarrow \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ to estimate the messages from its observations, $(\hat{w}_1, \ldots, \hat{w}_{M_k}) = D(Y)$. We say that the rate (per user) $R(H_N)$ is achievable if there exist encoders and a decoder such that, for any $\epsilon > 0$ and $T$ large enough, $\mathbb{P}( (\hat{w}_1, \ldots, \hat{w}_{M_k}) \neq (w_1, \ldots, w_{M_k}) ) \leq \epsilon$ so long as $R < R(H_N)$. For the static case, we will write $R(H)$ instead of $R(H_N)$. 

As noted earlier, the transmitters will not know the channel realization and thus we will have to tolerate some probability of outage. Assume that a given scheme achieves rate $R_{\text{scheme}}(H_N)$. For a target rate per user $R$, we define the outage probability of the scheme $P_{\text{outage}}(R) \triangleq \mathbb{P}(R_{\text{scheme}}(H_N) < R)$. Similarly, for a target outage probability $\rho \in (0, 1]$, the outage rate of the scheme is $R_{\text{outage}}(\rho) \triangleq \sup \{ R : P_{\text{outage}}(R) \leq \rho \}$. 

Theorem 1: For a target outage probability $\rho$, the optimal equalization matrix is given by

$H_{\text{opt}} = \min_{R \leq R_{\text{outage}}(\rho)} \frac{1}{2T} \log \det \left( I + \frac{SNR}{2} H^{T} H \right)$. 

Theorem 2: For a target outage probability $\rho$, the optimal equalization matrix is given by

$H_{\text{opt}} = \min_{R \leq R_{\text{outage}}(\rho)} \frac{1}{2T} \log \left( 1 + \frac{SNR}{\|b_m\|^2 + \sum_{i \neq m} (b_m^T h_i)^2} \right)$. 

Equation (3) is the minimum mean-square error (MMSE) projection.

$B_{\text{MMSE}} = SNR H^{T} (I + SNR HH^{T})^{-1}$. 

Equation (4) is a well-known result that the optimal equalization matrix is given by the minimum mean-square error (MMSE) projection.
B. Block Fading

1) Joint ML Receiver: As before, joint ML decoding is optimal and achieves the following rate

\[
R_{\text{ML}}(H_1^N) = \min_{S \subseteq \{1, \ldots, M_{\text{tx}}\}} \frac{1}{2N |S|} \sum_{n=1}^{N} \log \det \left( I + \text{SNR} H_n, S H_n^T \right)
\]

where \(H_n, S\) is the submatrix consisting of the columns of \(H_n\) whose indices are in the subset \(S\).

2) AM Linear Receiver: For each subblock, the receiver uses an equalization matrix \(B_n \in \mathbb{R}^{M_{\text{tx}} \times M_{\text{tx}}}\) to separate the data streams, which results in an effective subblock output \(Y_n = B_n Y_n\). The \(m^{th}\) row of each effective output is passed to an AM single-user decoder, which attempts to make an estimate of \(w_m\). The following rate is achievable:

\[
R_{\text{AM,Linear}}(H_1^N, B_1^N) = \min_{m=1, \ldots, M_{\text{tx}}} \frac{1}{2} \log \left( \frac{1}{N} \sum_{n=1}^{N} 1 + \text{SNR}_{n,m} \right)
\]

\[
\text{SNR}_{n,m} = \frac{\text{SNR}(b_{n,m}^T h_{n,m})^2}{\|b_{n,m}\|^2 + \text{SNR} \sum_{i \neq m} (b_{n,i}^T h_{n,i})^2}.
\]

where \(b_{n,m}^T\) is the \(m^{th}\) row of the equalization matrix \(B_n\) and \(h_{n,m}\) is the \(m^{th}\) column of the channel matrix \(H_n\). This can be proved as later in Section IV-B1, by plugging \(A = I\) in (16).

As before, the optimal equalization matrix for the \(n^{th}\) subblock is the MMSE projection

\[
B_{\text{MMSE},n} = \text{SNR} H_n^T (I + \text{SNR} H_n H_n^T)^{-1}.
\]

3) GM Linear Receiver: By using a more powerful single-user decoder, we can benefit from the fact that the effective noise is not i.i.d. across subblocks. We use the same equalization steps as in the AM case (and it also follows that MMSE equalization is optimal) and ultimately obtain the following achievable rate:

\[
R_{\text{GM,Linear}}(H_1^N, B_1^N) = \min_{m=1, \ldots, M_{\text{tx}}} \frac{1}{2N} \sum_{n=1}^{N} \log (1 + \text{SNR}_{n,m})
\]

where \(\text{SNR}_{n,m}\) is given in (6).

IV. INTEGER-FORCING RECEIVERS

The integer-forcing (IF) receiver exploits the fact that lattice codebooks are closed under integer-linear combinations, which implies that a single-user decoder can recover a linear combination of the messages (also known as compute-and-forward [8]). Specifically, the goal is first to recover \(M_{\text{tx}}\) linear combinations of the form \(u_m = \sum_{\ell=1}^{M_{\text{tx}}} a_{m,\ell} w_\ell \mod p\) where the \(a_{m,\ell} \in \mathbb{Z}\) are integer coefficients. If the integer matrix \(A = (a_{m,\ell})\) is full rank modulo \(p\), then the receiver can solve for the original messages. See [1], [8] for an in-depth discussion.

A. Static Fading

We briefly review the IF receiver introduced in [1] for the static case. After applying an equalization matrix \(B \in \mathbb{R}^{M_{\text{tx}} \times M_{\text{tx}}}\), the effective output passed to the \(m^{th}\) single-user decoder is

\[
\tilde{y}_m = b_m^T Y = \frac{a_m^T X}{\text{SNR}} + (b_m^T H - a_m^T) X + b_m^T Z.
\]

where \(a_m^T\) and \(b_m^T\) are the \(m^{th}\) rows of \(A\) and \(B\), respectively. The \(m^{th}\) decoder uses \(\tilde{y}_m\) to make an estimate of the linear combination \(u_m = \sum_{\ell=1}^{M_{\text{tx}}} a_{m,\ell} w_\ell \mod p\). Assuming these estimates are correct, the receiver solves the corresponding system of linear equations to recover the original messages. Define \(\log^+(x) \triangleq \max(0, \log(x))\). As shown in [1], the following rate is achievable for integer-forcing:

\[
R_{\text{IF}}(H, B) = \max_{A \in \mathbb{Z}^{M_{\text{tx}} \times M_{\text{tx}}}} \min_{B \in \mathbb{R}^{M_{\text{tx}} \times M_{\text{tx}}}} \frac{1}{2} \log \left( \frac{\text{SNR}}{\sigma_{\text{eff},m}(H, B)} \right)
\]

where \(\sigma_{\text{eff},m}(H, B) = \|b_m^T\|^2 + \text{SNR} \|b_m^T H - a_m^T\|^2\) is the effective noise variance encountered in decoding the \(m^{th}\) linear combination. Plugging in the optimal MMSE equalization matrix \(B_{\text{MMSE}} = \text{SNR} A H^T (I + \text{SNR} H H^T)^{-1}\) and applying Woodbury’s matrix identity, we can rewrite (10) as

\[
R_{\text{IF}}(H) = \max_{A \in \mathbb{Z}^{M_{\text{tx}} \times M_{\text{tx}}}} \min_{B \in \mathbb{R}^{M_{\text{tx}} \times M_{\text{tx}}}} \frac{1}{2} \log \left( \frac{\text{SNR}}{\|B_{\text{MMSE}}\|} \right),
\]

where \(F = (\text{SNR}^{-1} I + H^T H)^{-1/2}\). It follows that the problem of maximizing the achievable rate is equivalent to the problem of finding the successive minima of the lattice induced by \(F\). Although finding the optimal integer vectors is a challenging problem, the LLL algorithm [9] can be used to find near-optimal solutions in polynomial time.

B. Block Fading

We now propose a class of IF receivers for the block fading case. See Figure 1 for a block diagram. For \(n = 1, \ldots, N\), the receiver applies an equalization matrix \(B_n \in \mathbb{R}^{M_{\text{tx}} \times M_{\text{tx}}}\), to obtain the effective channel outputs

\[
\tilde{y}_{n,m} = b_{n,m}^T Y_n
\]

\[
a_{n,m}^T X_n + z_{\text{eff},n,m}
\]

\[
z_{\text{eff},n,m} = (b_{n,m}^T H_n - a_{n,m}^T) X_n + b_{n,m}^T Z
\]

where \(b_{n,m}^T\) is the \(m^{th}\) row of the equalization matrix \(B_n\) and \(z_{\text{eff},n,m}\) represents the effective noise encountered by the \(m^{th}\) decoder during the \(n^{th}\) subblock. Notice that \(a_{n,m}\) is fixed throughout the block. Ideally, we would like to adapt \(a_{n,m}\) to match the equalized channel \(b_{n,m}^T H_n\), but this will destroy the closure property of the underlying lattice codebook. It can be argued that the effective variance of \(z_{\text{eff},n,m}\) is \(\sigma_{\text{eff},n,m}^2(B_n, H_n) = \|b_{n,m}^T\|^2 + \text{SNR} \|b_{n,m}^T H_n - a_{n,m}^T\|^2\).
Collecting the effective channel outputs across subblocks, we can write the effective channel seen by the $m^{th}$ decoder as

$$\tilde{y}_m = a_m^T X + z_{\text{eff},m}$$  \hspace{1cm} (15)$$

where $z_{\text{eff},m} = [z_{\text{eff},1,m} \cdots z_{\text{eff},N,m}]$.

We now state the achievable rates of IF via both AM and GM decoding.

1) AM IF Receiver:

**Theorem 1:** The following rate is achievable via an AM IF receiver:

$$R_{\text{AM,IF}}(H_1^N) = \max_{A \in Z_{M_{\text{Tx}} \times M_{\text{Rx}}}^{N \times M_{\text{Tx}}}, m = 1, \ldots, M_{\text{Rx}}} \min_{\text{rank}(A) = M_{\text{Rx}}} \frac{1}{2N} \frac{1}{2} \log_2 \left( \frac{\text{SNR}}{\|F_n a_m\|^2} \right).$$

where $F_{\text{eq}}$ is obtained by factoring $F_{\text{eq}}F_{\text{eq}}^T = \frac{1}{N} \sum_{n=1}^{N} F_nF_n^T$ where $F_n = (SNR^{-1}I + H_1^T H_n)^{-1/2}$.

**Proof:** The average effective variance across subblocks is

$$\sigma_{\text{eff,AM},m} = \frac{1}{N} \sum_{n=1}^{N} \sigma_{\text{eff,n,m}}^2 (B_n, H_n) = \frac{1}{N} \sum_{n=1}^{N} \|F_n a_m\|^2 = \|F_{\text{eq}} a_m\|^2$$

where $(\text{a})$ follows from plugging in the optimal MMSE equalization matrix $B_{\text{MMSE,n}} = \text{SNR} A H_n^T (I + \text{SNR} H_n H_n^T)^{-1}$ and applying Woodbury's matrix identity. It is straightforward to show that, by employing nested lattice codebooks that are good for semi-spherical noise [10, Definition 7.8.2], the stated rate is achievable. See [10] for a detailed discussion of the codebook construction and achievability proof.

Notice that, like the static case, we can find near-optimal integer vectors by applying the LLL algorithm to the lattice induced by $F_{\text{eq}}$.

2) GM IF Receiver:

**Theorem 2:** The following rate is achievable via a GM IF receiver:

$$R_{\text{GM,IF}}(H_1^N) = \max_{A \in Z_{M_{\text{Tx}} \times M_{\text{Rx}}}^{N \times M_{\text{Tx}}}, \text{rank}(A) = M_{\text{Rx}}} \min_{m} \frac{1}{2N} \sum_{n=1}^{N} \log_2 \left( \frac{\text{SNR}}{\|F_n a_m\|^2} \right)$$

where $F_n = (SNR^{-1}I + H_1^T H_n)^{-1/2}$.

**Proof:** See the Appendix.

Note that the AM of each effective noise variance is greater than its GM. Therefore, the rate of the GM IF receiver at least as high as the rate of the AM IF receiver. Unfortunately, the problem of finding the optimal integer vectors to optimize the GM IF rate expression does not directly correspond to finding the successive minima of a lattice. It remains an open problem to find a polynomial-time algorithm that locates near-optimal integer vectors. For our simulations, we employ an exhaustive search algorithm over all integer matrices with bounded entries.

For our simulations, we set the number of subblocks to $N = 8$, and drew each channel matrix $H_n$ according to an i.i.d. Gaussian distribution. Our plots are generated from 10000 realizations.

In Figure 2, we consider the performance of both, AM and GM, receivers with $M_{\text{Tx}} = 2$ transmitters and $M_{\text{Rx}} = 2$ receive antennas in the case of 0.1 outage probability. We plotted the performance of capacity-achieving joint ML decoding, IF decoding, and MMSE decoding. Note that IF decoding is nearly the same as MMSE decoding. Finally, we also included curves for successive integer-forcing (SIF) [3]. Recall that this refers to the technique of using decoded linear combinations to reduce the effective noise encountered in subsequent decoding steps. The static case first appeared in [3] and the AM SIF case follows similarly. Note that AM and GM SIF decoding outperforms both AM and GM MMSE significantly.

In Figure 3, we plotted the CDF of the outage rate at a fixed SNR point ($\text{SNR} = 25$ dB). Note that, for the AM case, the difference in performance between the IF and MMSE receivers is more pronounced in the moderate outage probability regime. In Figure 4, we compare both, the IF and SIF, receivers in both fading cases. The remarkable difference between AM IF and AM SIF receivers, relative to the static case, emphasizes the importance of successive cancellation for IF receivers in the block fading case.

Finally, Figure 5 shows the performance of AM linear and IF receivers for $M_{\text{Tx}} = 3$ transmitters and $M_{\text{Rx}} = 3$ receivers in the case of 0.1 outage probability.

**V. Simulation Results**

We proposed two receivers for integer-forcing for the block fading channel model, which outperform conventional linear receivers. One open question is whether one can connect the achievable rates of the GM IF receiver to a fundamental lattice quantity, such as the successive minima, in order to develop analytical bounds as in [3]. Another interesting problem is to

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$^4$See [1, Lemma 2] for details on how to set this bound.
develop a lattice reduction algorithm for finding the optimal integers for the GM IF receivers.

**APPENDIX: ACHIEVABILITY PROOF FOR GM IF**

Due to space limitations, we only provide a sketch for the proof. Specifically, consider the following effective channel model: $y = \lambda + z$ where $\lambda$ is a lattice codeword of length $T$, $z = [z_1 \cdots z_N]^T$, and each $z_n$ is a length-$(T/N)$ i.i.d. Gaussian vector with mean 0 and variance $\sigma_n^2$. The main difficulty is that the noise variances $\sigma_1^2, \ldots, \sigma_N^2$ are not known a priori. Specifically, each noise variance is arbitrarily chosen from the interval $[0, \sigma_{\text{max}}^2]$. (In our problem setting, we can choose $\sigma_{\text{max}}^2$ such that the probability that the effective noise variances fall outside the interval has a negligible impact on the outage probability.)

As a first step, we quantize the interval $[0, \sigma_{\text{max}}^2]$ to precision $\delta$. Let $\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_N^2$ denote noise variances after quantization. For each of the $(2\delta N)^N$ possible quantized noise variances, we select an ambiguity decoder following the methodology in [11], [12]. For any $0 < \sigma_{\text{GM}}^2 < \text{SNR}$, it can be shown that the average probability of error across the standard Construction A ensemble vanishes exponentially fast so long as $(\prod_{n=1}^N \tilde{\sigma}_n^2)^{1/N} + N\delta < \sigma_{\text{GM}}^2$. We can now apply the union bound to show that the average probability of error vanishes exponentially fast so long as $(\prod_{n=1}^N \tilde{\sigma}_n^2)^{1/N} + N\delta < \sigma_{\text{GM}}^2$.

Finally, it can be argued that there exists one good sequence of nested lattice codebooks with rate approaching

$$\frac{1}{2} \log \left( \frac{\text{SNR}}{\sigma_{\text{GM}}^2} \right)$$

so long as $(\prod_{n=1}^N \tilde{\sigma}_n^2)^{1/N} < \sigma_{\text{GM}}^2$.

**REFERENCES**


