The Impact of Channel Variation on Integer-Forcing Receivers

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***Abstract*—Consider several single-antenna transmitters that wish to simultaneously communicate with a multiple-antenna receiver. Recent work has proposed the integer-forcing linear receiver architecture as an alternative to conventional linear receivers. The key idea is to first use linear equalization to obtain an integer-valued effective channel, then employ single-user decoders to recover integer-linear combinations of the messages, and finally solve these for the desired messages. For the special case where the channel matrix remains fixed for the duration of the codeword, it has been shown that integer-forcing can operate very close to the performance of optimal joint maximum likelihood decoding. In this paper, we investigate the impact of channel variation on the integer-forcing linear receiver and show it still retains an advantage over conventional linear receivers, despite the fact that the integer coefficients must remain fixed across the codeword duration.**

I. Introduction

Linear receivers are often employed as a means of reducing the implementation complexity of multiple-input multiple-output (MIMO) decoding. The basic idea is to first separate the data streams via linear equalization and then recover them via single-user decoding. However, in many scenarios, conventional linear receivers fall short of the performance of optimal joint maximum likelihood (ML) decoding of the data streams. Recent work [1] has proposed a variation on linear receivers known as *integer-forcing* that can operate much closer to the joint ML performance. The underlying idea is to use the single-user decoders to recover integer-linear combinations of the messages and then solve these for the original messages. Several interesting extensions include integer-forcing beamforming [2], successive integer-forcing [3], and space-time coding at the transmitters [4].

All of the above works make the tacit assumption that the channel remains fixed for the duration of the codeword. However, in many important scenarios, the channel may vary significantly within the span of a single codeword. For instance, under orthogonal frequency-division multiplexing (OFDM), channel variation will occur across OFDM symbols if the channel is frequency-selective [5]. At a first glance, it may seem that channel variation precludes the possibility of integer-forcing, since lattice codebooks are only closed under integer-linear combinations, i.e., the integer coefficients cannot be adjusted to track the channel variation. As we will demonstrate, the associated rate loss can be mitigated by allowing the linear equalizer to track the channel and by employing single-user decoders that can take advantage of non-spherical noise.

We will focus on the important special case of block fading [6], which can be viewed as a simple model for the fading that results from employing OFDM over frequency-selective channels. As in [1], we assume that we only have channel state information at the receiver (CSIR). To ensure a fair comparison between conventional and integer-forcing linear receivers, we will consider two classes of single-user decoders: "arithmetic mean" decoders that succeed only if the noise falls within a sphere of a predetermined radius and "geometric mean" decoders that can take advantage of the reduced effective volume of ellipsoidal noise[1].

II. Problem Statement

We will denote column vectors by boldface lowercase (e.g., $\mathbf{x}$) and matrices by boldface uppercase (e.g., $\mathbf{X}$). Let $\mathbf{X}^{\mathsf{T}}$ denote the transpose of a matrix $\mathbf{X}$ and let $\mathbf{X}_1^N \triangleq (\mathbf{X}_1, \dots, \mathbf{X}_N)$ denote a sequence of matrices. For simplicity, we focus on real-valued channels and note that complex-valued channels can be handled via their real-valued decomposition [1].

There are M_{Tx} single-antenna transmitters that communicate to a receiver equipped with M_{Rx} antennas. The ℓ^{th} transmitter has a *message* $\mathbf{w}_\ell \in \mathbb{Z}_p^k$ where p is prime. Using an encoder $\mathcal{E}_\ell : \mathbb{Z}_p^k \to \mathbb{R}^T$, it maps its message into a sequence of channel inputs, $(x_\ell[1], \dots, x_\ell[T]) = \mathcal{E}_\ell(\mathbf{w}_\ell)$ where T is the coding blocklength. Each transmitter's input must satisfy the usual power constraint, $\frac{1}{T}\sum_{t=1}^{T}(x_\ell[t])^2 \leq \mathsf{SNR}$. The rate of each message is $R = \frac{k}{T}\log p$.

At time t, the receiver observes

$$\mathbf{y}[t] = \mathbf{H}[t]\mathbf{x}[t] + \mathbf{z}[t]$$

where $\mathbf{H}[t] \in \mathbb{R}^{M_{\text{Rx}} \times M_{\text{Tx}}}$ is the channel matrix, $\mathbf{x}[t] \triangleq \begin{bmatrix} x_1[t] & \cdots & x_{M_{\text{Tx}}}[t] \end{bmatrix}^{\mathsf{T}}$ is the vector of channel inputs, and $\mathbf{z}[t] \in \mathbb{R}^{M_{\text{Rx}}}$ is independent and identically distributed (i.i.d.) Gaussian noise, $\mathbf{z}[t] \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$.

We consider two special cases of channel matrix fading distributions: static and block fading. In both cases, we assume that only CSIR is available (although the transmitters are aware of the fading statistics).

[1] In practice, it may be possible to implement this decoder via spatially-coupled low-density parity-check (LDPC) coding [7].

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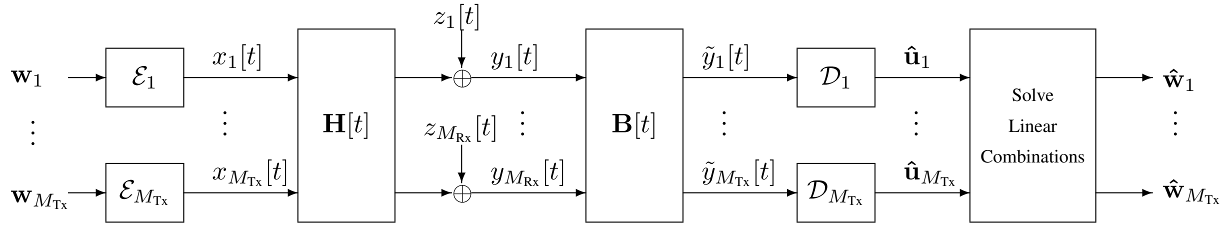

Fig. 1. Block diagram of the integer-forcing linear receiver for time-varying MIMO channels.

Static Channel Model: In this model, we assume that a channel matrix $\mathbf{H}$ is randomly generated at the beginning and then stays fixed for the remaining T time slots, $\mathbf{H}[t] = \mathbf{H}$. Collecting the receiver's observations into a matrix $\mathbf{Y} \triangleq \big[\mathbf{y}[1] \ \cdots \ \mathbf{y}[T]\big]$, we get

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{Z}, \tag{1}$$

where $\mathbf{X} \triangleq \big[\mathbf{x}[1] \ \cdots \ \mathbf{x}[T]\big]$ and $\mathbf{Z} \triangleq \big[\mathbf{z}[1] \ \cdots \ \mathbf{z}[T]\big]$. We introduce this model mainly to compare our rate expressions and simulations to those obtained in prior work [1], [3], [4].

Block Fading Channel Model: This model captures channel variation by assuming that the channel matrix takes on N different realizations over the duration of the codeword. Specifically, say that the matrices $\mathbf{H}_1, \ldots, \mathbf{H}_N$ are generated i.i.d. according to some distribution[2]. These matrices represent the channel realization during the n^{th} subblock for $n = 1, \ldots, N$. Specifically, for $t = (n-1)\lfloor T/N \rfloor + 1, \ldots, n\lfloor T/N \rfloor$, the channel matrix is $\mathbf{H}[t] = \mathbf{H}_n$.

For notational convenience, we define

$$\begin{aligned}
\mathbf{X}_n &\triangleq \Big[\mathbf{x}\big[(n-1)\lfloor T/N \rfloor + 1\big] \ \cdots \ \mathbf{x}\big[n\lfloor T/N \rfloor\big]\Big] \\
\mathbf{Y}_n &\triangleq \Big[\mathbf{y}\big[(n-1)\lfloor T/N \rfloor + 1\big] \ \cdots \ \mathbf{y}\big[n\lfloor T/N \rfloor\big]\Big] \\
\mathbf{Z}_n &\triangleq \Big[\mathbf{z}\big[(n-1)\lfloor T/N \rfloor + 1\big] \ \cdots \ \mathbf{z}\big[n\lfloor T/N \rfloor\big]\Big] \ .
\end{aligned}$$

This permits us to express the input-output relationship during the n^{th} subblock as

$$\mathbf{Y}_n = \mathbf{H}_n \mathbf{X}_n + \mathbf{Z}_n \ . \tag{2}$$

The receiver uses a decoder $\mathcal{D} : \mathbb{R}^{M_{\text{Rx}} \times T} \to \mathbb{Z}_p^k \times \cdots \times \mathbb{Z}_p^k$ to obtain estimates of the messages from its observations, $(\hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_{M_{\text{Tx}}}) = \mathcal{D}(\mathbf{Y})$. We say that the rate (per user) $R(\mathbf{H}_1^N)$ is achievable if there exist encoders and a decoder such that, for any $\epsilon > 0$ and T large enough, $\mathbb{P}\left((\hat{\mathbf{w}}_1, \ldots, \hat{\mathbf{w}}_{M_{\text{Tx}}}) \neq (\mathbf{w}_1, \ldots, \mathbf{w}_{M_{\text{Tx}}})\right) \leq \epsilon$ so long as $R < R(\mathbf{H}_1^N)$. For the static case, we will write $R(\mathbf{H})$ instead of $R(\mathbf{H}_1^N)$.

As noted earlier, the transmitters will not know the channel realization and thus we will have to tolerate some probability of outage. Assume that a given scheme achieves rate $R_{\text{scheme}}(\mathbf{H}_1^N)$. For a target rate per user R, we define the *outage probability* of the scheme $p_{\text{outage}}(R) \triangleq \mathbb{P}\big(R_{\text{scheme}}(\mathbf{H}_1^N) < R\big)$. Similarly, for a target outage probability $\rho \in (0, 1]$, the *outage rate* of the scheme is $R_{\text{outage}}(\rho) \triangleq \sup\big\{R : p_{\text{outage}}(R) \leq \rho\big\}$.

[2]We consider i.i.d. subblocks for simplicity. For non-i.i.d. subblocks, we should employ power allocation across subblocks using the transmitters' knowledge of the fading statistics.

III. Conventional Receiver Architectures

In this section, we give a brief summary of conventional MIMO receiver architectures. We begin by reviewing the static case and then move onto the block fading case. For linear receivers, we consider two special cases of single-user decoding. The first class of arithmetic mean (AM) decoders succeed only if the effective noise observed at the decoder falls within a sphere of a prespecified radius[3]. The second class of geometric mean (GM) decoders are able to account for the time-varying nature of the effective noise variance, i.e., they succeed if the differential entropy of the effective noise falls below a certain threshold. (This can be inferred from the geometric mean of the effective noise variance across subblocks, hence the name.) Throughout this section, we assume that the transmitters employ i.i.d. Gaussian codebooks.

A. Static Fading

1) Joint ML Receiver: If we employ joint ML decoding at the receiver, the following rate is achievable

$$R_{\text{ML}}(\mathbf{H}) = \min_{\mathcal{S} \subseteq \{1, \ldots, M_{\text{Tx}}\}} \frac{1}{2|\mathcal{S}|} \log\det\left(\mathbf{I} + \mathsf{SNR}\, \mathbf{H}_{\mathcal{S}} \mathbf{H}_{\mathcal{S}}^{\mathsf{T}}\right) \ .$$

where $\mathbf{H}_{\mathcal{S}}$ is the submatrix consisting of the columns of $\mathbf{H}$ whose indices are in the subset $\mathcal{S}$.

2) Linear Receiver: For static fading, AM and GM linear receivers can attain the same rates. Prior to decoding, the receiver uses an equalization matrix $\mathbf{B} \in \mathbb{R}^{M_{\text{Tx}} \times M_{\text{Rx}}}$ to separate the data streams and obtain the effective channel output $\tilde{\mathbf{Y}} = \mathbf{B}\mathbf{Y}$. The m^{th} row of $\tilde{\mathbf{Y}}$ is sent to a single-user decoder $\mathcal{D}_m : \mathbb{R}^T \to \mathbb{Z}_p^k$ that attempts to recover the m^{th} message $\mathbf{w}_m$. This yields the following achievable rate:

$$\begin{aligned}
&R_{\text{Linear}}(\mathbf{H}, \mathbf{B}) \\
&= \min_{m=1,\ldots,M_{\text{Tx}}} \frac{1}{2} \log\left(1 + \frac{\mathsf{SNR}(\mathbf{b}_m^{\mathsf{T}} \mathbf{h}_m)^2}{\|\mathbf{b}_m\|^2 + \mathsf{SNR} \sum_{i \neq m} (\mathbf{b}_m^{\mathsf{T}} \mathbf{h}_i)^2}\right)
\end{aligned} \tag{3}$$

where $\mathbf{b}_m^{\mathsf{T}}$ is the m^{th} row of the equalization matrix $\mathbf{B}$ and $\mathbf{h}_m$ is the m^{th} column of the channel matrix $\mathbf{H}$. It is well-known that the optimal equalization matrix is given by the minimum mean-squared error (MMSE) projection,

$$\mathbf{B}_{\text{MMSE}} = \mathsf{SNR}\, \mathbf{H}^{\mathsf{T}} \big(\mathbf{I} + \mathsf{SNR}\, \mathbf{H}\mathbf{H}^{\mathsf{T}}\big)^{-1} \ . \tag{4}$$

[3]One can alternatively view the arithmetic mean decoder as making the worst-case assumption that the effective noise is i.i.d. Gaussian.

B. Block Fading

1) Joint ML Receiver: As before, joint ML decoding is optimal and achieves the following rate

$$R_{\text{ML}}(\mathbf{H}_1^N) \tag{5}$$
$$= \min_{\mathcal{S}\subseteq\{1,\dots,M_{\text{Tx}}\}} \frac{1}{2N|\mathcal{S}|}\sum_{n=1}^{N}\log\det\left(\mathbf{I}+\mathsf{SNR}\,\mathbf{H}_{n,\mathcal{S}}\mathbf{H}_{n,\mathcal{S}}^{\mathsf{T}}\right)$$

where $\mathbf{H}_{n,\mathcal{S}}$ is the submatrix consisting of the columns of $\mathbf{H}_n$ whose indices are in the subset $\mathcal{S}$.

2) AM Linear Receiver: For each subblock, the receiver uses an equalization matrix $\mathbf{B}_n \in \mathbb{R}^{M_{\text{Tx}}\times M_{\text{Rx}}}$ to separate the data streams, which results in an effective subblock output $\tilde{\mathbf{Y}}_n = \mathbf{B}_n\mathbf{Y}_n$. The m^{th} row of each effective output is passed to an AM single-user decoder, which attempts to make an estimate of $\mathbf{w}_m$. The following rate is achievable:

$$R_{\text{AM,Linear}}(\mathbf{H}_1^N,\mathbf{B}_1^N) = \min_{m=1,\dots,M_{\text{Tx}}}\frac{1}{2}\log\left(\frac{1}{\frac{1}{N}\sum\limits_{n=1}^{N}\frac{1}{1+\mathsf{SINR}_{n,m}}}\right)$$

$$\mathsf{SINR}_{n,m} = \frac{\mathsf{SNR}(\mathbf{b}_{n,m}^{\mathsf{T}}\mathbf{h}_{n,m})^2}{\|\mathbf{b}_{n,m}\|^2+\mathsf{SNR}\sum\limits_{i\neq m}(\mathbf{b}_{n,m}^{\mathsf{T}}\mathbf{h}_{n,i})^2}\ . \tag{6}$$

where $\mathbf{b}_{n,m}^{\mathsf{T}}$ is the m^{th} row of the equalization matrix $\mathbf{B}_n$ and $\mathbf{h}_{n,m}$ is the m^{th} column of the channel matrix $\mathbf{H}_n$. This can be proved as later in SectionIV-B1, by plugging $\mathbf{A}=\mathbf{I}$ in (16).

As before, the optimal equalization matrix for the n^{th} subblock is the MMSE projection

$$\mathbf{B}_{\text{MMSE},n} = \mathsf{SNR}\,\mathbf{H}_n^{\mathsf{T}}\left(\mathbf{I}+\mathsf{SNR}\,\mathbf{H}_n\mathbf{H}_n^{\mathsf{T}}\right)^{-1}\ . \tag{7}$$

3) GM Linear Receiver: By using a more powerful single-user decoder, we can benefit from the fact that the effective noise is not i.i.d. across subblocks. We use the same equalization steps as in the AM case (and it also follows that MMSE equalization is optimal) and ultimately obtain the following achievable rate:

$$R_{\text{GM,Linear}}(\mathbf{H}_1^N,\mathbf{B}_1^N) = \min_m \frac{1}{2N}\sum_{n=1}^{N}\log\left(1+\mathsf{SINR}_{n,m}\right) \tag{8}$$

where $\mathsf{SINR}_{n,m}$ is given in (6).

IV. Integer-Forcing Receivers

The integer-forcing (IF) receiver exploits the fact that lattice codebooks are closed under integer-linear combinations, which implies that a single-user decoder can recover a linear combination of the messages (also known as compute-and-forward [8]). Specifically, the goal is first to recover M_{Tx} linear combinations of the form $\mathbf{u}_m = \left[\sum_{\ell=1}^{M_{\text{Tx}}} a_{m,\ell}\mathbf{w}_\ell\right] \bmod p$ where the $a_{m,\ell}\in\mathbb{Z}$ are integer coefficients. If the integer matrix $\mathbf{A}=\{a_{m,\ell}\}$ is full rank modulo p, then the receiver can solve for the original messages. See [1], [8] for an in-depth discussion.

A. Static Fading

We briefly review the IF receiver introduced in [1] for the static case. After applying an equalization matrix $\mathbf{B}\in\mathbb{R}^{M_{\text{Tx}}\times M_{\text{Rx}}}$, the effective output passed to the m^{th} single-user decoder is

$$\tilde{\mathbf{y}}_m = \mathbf{b}_m^{\mathsf{T}}\mathbf{Y} = \underbrace{\mathbf{a}_m^{\mathsf{T}}\mathbf{X}}_{\text{lattice codeword}} + \underbrace{\left(\mathbf{b}_m^{\mathsf{T}}\mathbf{H}-\mathbf{a}_m^{\mathsf{T}}\right)\mathbf{X}+\mathbf{b}_m^{\mathsf{T}}\mathbf{Z}}_{\text{effective noise}}\ . \tag{9}$$

where $\mathbf{a}_m^{\mathsf{T}}$ and $\mathbf{b}_m^{\mathsf{T}}$ are the m^{th} rows of $\mathbf{A}$ and $\mathbf{B}$, respectively. The m^{th} decoder uses $\tilde{\mathbf{y}}_m$ to make an estimate of the linear combination $\mathbf{u}_m = \left[\sum_{\ell=1}^{M_{\text{Tx}}} a_{m,\ell}\mathbf{w}_\ell\right] \bmod p$. Assuming these estimates are correct, the receiver solves the corresponding system of linear equations to recover the original messages.

Define $\log^+(x) \triangleq \max(0,\log(x))$.As shown in [1], the following rate is achievable for integer-forcing:

$$R_{\text{IF}}(\mathbf{H},\mathbf{B}) \tag{10}$$
$$= \max_{\substack{\mathbf{A}\in\mathbb{Z}^{M_{\text{Tx}}\times M_{\text{Tx}}}\\ \text{rank}(\mathbf{A})=M_{\text{Tx}}}}\ \max_{\mathbf{B}\in\mathbb{R}^{M_{\text{Tx}}\times M_{\text{Rx}}}}\ \min_m \frac{1}{2}\log^+\left(\frac{\mathsf{SNR}}{\sigma_{\text{eff},m}^2(\mathbf{H},\mathbf{B})}\right)$$

where $\sigma_{\text{eff},m}^2(\mathbf{H},\mathbf{B}) = \|\mathbf{b}_m^{\mathsf{T}}\|^2+\mathsf{SNR}\|\mathbf{b}_m^{\mathsf{T}}\mathbf{H}-\mathbf{a}_m^{\mathsf{T}}\|^2$ is the effective noise variance encountered in decoding the m^{th} linear combination. Plugging in the optimal MMSE equalization matrix $\mathbf{B}_{\text{MMSE}} = \mathsf{SNR}\,\mathbf{A}\mathbf{H}^{\mathsf{T}}\left(\mathbf{I}+\mathsf{SNR}\,\mathbf{H}\mathbf{H}^{\mathsf{T}}\right)^{-1}$ and applying Woodbury's matrix identity, we can rewrite (10) as

$$R_{\text{IF}}(\mathbf{H}) = \max_{\substack{\mathbf{A}\in\mathbb{Z}^{M_{\text{Tx}}\times M_{\text{Tx}}}\\ \text{rank}(\mathbf{A})=M_{\text{Tx}}}}\ \min_{m=1,\dots,M_{\text{Tx}}}\frac{1}{2}\log^+\left(\frac{\mathsf{SNR}}{\|\mathbf{F}\mathbf{a}_m\|^2}\right), \tag{11}$$

where $\mathbf{F} = (\mathsf{SNR}^{-1}\mathbf{I}+\mathbf{H}^{\mathsf{T}}\mathbf{H})^{-1/2}$. It follows that the problem of maximizing the achievable rate is equivalent to the problem of finding the successive minima of the lattice induced by $\mathbf{F}$. Although finding the optimal integer vectors is a challenging problem, the LLL algorithm [9] can be used to find near-optimal solutions in polynomial time.

B. Block Fading

We now propose a class of IF receivers for the block fading case. See Figure 1 for a block diagram. For $n=1,\dots,N$, the receiver applies an equalization matrix $\mathbf{B}_n\in\mathbb{R}^{M_{\text{Tx}}\times M_{\text{Rx}}}$, to obtain the effective channel outputs

$$\tilde{\mathbf{y}}_{n,m} = \mathbf{b}_{n,m}^{\mathsf{T}}\mathbf{Y}_n \tag{12}$$
$$= \mathbf{a}_m^{\mathsf{T}}\mathbf{X}_n+\mathbf{z}_{\text{eff},n,m} \tag{13}$$
$$\mathbf{z}_{\text{eff},n,m} = \left(\mathbf{b}_{n,m}^{\mathsf{T}}\mathbf{H}_n-\mathbf{a}_m^{\mathsf{T}}\right)\mathbf{X}_n+\mathbf{b}_{n,m}^{\mathsf{T}}\mathbf{Z}_n \tag{14}$$

where $\mathbf{b}_{n,m}^{\mathsf{T}}$ is the m^{th} row of the equalization matrix $\mathbf{B}_n$ and $\mathbf{z}_{\text{eff},n,m}$ represents the effective noise encountered by the m^{th} decoder during the n^{th} subblock. Notice that $\mathbf{a}_m$ is fixed throughout the block. Ideally, we would like to adapt $\mathbf{a}_m$ to match the equalized channel $\mathbf{b}_{n,m}^{\mathsf{T}}\mathbf{H}_n$ but this will destroy the closure property of the underlying lattice codebook. It can be argued that the effective variance of $\mathbf{z}_{\text{eff},n,m}$ is $\sigma_{\text{eff},n,m}^2(\mathbf{B}_n,\mathbf{H}_n) = \|\mathbf{b}_{n,m}^{\mathsf{T}}\|^2+\mathsf{SNR}\|\mathbf{b}_{n,m}^{\mathsf{T}}\mathbf{H}_n-\mathbf{a}_m^{\mathsf{T}}\|^2$.

Collecting the effective channel outputs across subblocks, we can write the effective channel seen by the m^{th} decoder as

$$\tilde{\mathbf{y}}_m = \mathbf{a}_m^{\mathsf{T}}\mathbf{X} + \mathbf{z}_{\text{eff},m} \tag{15}$$

where $\mathbf{z}_{\text{eff},m} = [\mathbf{z}_{\text{eff},1,m} \ \cdots \ \mathbf{z}_{\text{eff},N,m}]$.

We now state the achievable rates of IF via both AM and GM decoding.

1) AM IF Receiver:

Theorem 1: The following rate is achievable via an AM IF receiver:

$$R_{\text{AM,IF}}(\mathbf{H}_1^N) = \max_{\substack{\mathbf{A}\in\mathbb{Z}^{M_{\text{Tx}}\times M_{\text{Tx}}} \\ \text{rank}(\mathbf{A})=M_{\text{Tx}}}} \ \min_{m=1,\ldots,M_{\text{Tx}}} \frac{1}{2}\log^+\left(\frac{\mathsf{SNR}}{\|\mathbf{F}_{\text{eq}}\mathbf{a}_m\|^2}\right).$$

where $\mathbf{F}_{\text{eq}}$ is obtained by factoring $\mathbf{F}_{\text{eq}}^{\mathsf{T}}\mathbf{F}_{\text{eq}} = \frac{1}{N}\sum_{n=1}^{N}\mathbf{F}_n^{\mathsf{T}}\mathbf{F}_n$ where $\mathbf{F}_n = (\mathsf{SNR}^{-1}\mathbf{I} + \mathbf{H}_n^{\mathsf{T}}\mathbf{H}_n)^{-1/2}$.

Proof: The average effective variance across subblocks is

$$\begin{aligned}\sigma^2_{\text{eff,AM},m} &= \frac{1}{N}\sum_{n=1}^{N}\sigma^2_{\text{eff},n,m}(\mathbf{B}_n,\mathbf{H}_n) \\ &\overset{(a)}{=} \frac{1}{N}\sum_{n=1}^{N}\|\mathbf{F}_n\mathbf{a}_m\|^2 = \|\mathbf{F}_{\text{eq}}\mathbf{a}_m\|^2\end{aligned} \tag{16}$$

where (a) follows from plugging in the optimal MMSE equalization matrix $\mathbf{B}_{\text{MMSE},n} = \mathsf{SNR}\,\mathbf{A}\mathbf{H}_n^{\mathsf{T}}\left(\mathbf{I}+\mathsf{SNR}\,\mathbf{H}_n\mathbf{H}_n^{\mathsf{T}}\right)^{-1}$ and applying Woodbury's matrix identity. It is straightforward to show that, by employing nested lattice codebooks that are good for semi-spherical noise [10, Definition 7.8.2], the stated rate is achievable. See [10] for a detailed discussion of the codebook construction and achievability proof. ■

Notice that, like the static case, we can find near-optimal integer vectors by applying the LLL algorithm to the lattice induced by $\mathbf{F}_{\text{eq}}$.

2) GM IF Receiver:

Theorem 2: The following rate is achievable via a GM IF receiver:

$$R_{\text{GM,IF}}(\mathbf{H}_1^N) = \max_{\substack{\mathbf{A}\in\mathbb{Z}^{M_{\text{Tx}}\times M_{\text{Tx}}} \\ \text{rank}(\mathbf{A})=M_{\text{Tx}}}} \ \min_{m} \frac{1}{2N}\sum_{n=1}^{N}\log\left(\frac{\mathsf{SNR}}{\|\mathbf{F}_n\mathbf{a}_m\|^2}\right)$$

where $\mathbf{F}_n = (\mathsf{SNR}^{-1}\mathbf{I} + \mathbf{H}_n^{\mathsf{T}}\mathbf{H}_n)^{-1/2}$.

Proof: See the Appendix. ■

Note that the AM of each effective noise variance is greater than its GM. Therefore, the rate of the GM IF receiver at least as high as the rate of the AM IF receiver. Unfortunately, the problem of finding the optimal integer vectors to optimize the GM IF rate expression does not directly correspond to finding the successive minima of a lattice. It remains an open problem to find a polynomial-time algorithm that locates near-optimal integer vectors. For our simulations, we employ an exhaustive search algorithm over all integer matrices with bounded entries[4].

[4]See [1, Lemma 2] for details on how to set this bound.

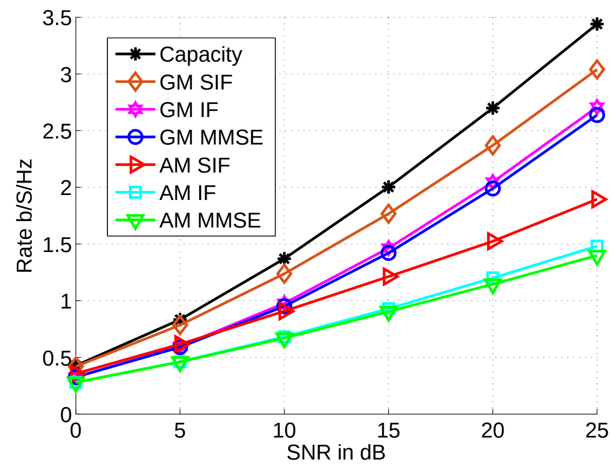

Fig. 2. Comparison of linear and IF receivers at outage probability $\rho = 0.1$ over $N = 8$ subblocks with $M_{\text{Tx}} = 2$ transmitters and $M_{\text{Rx}} = 2$ receive antennas.

V. Simulation Results

For our simulations, we set the number of subblocks to $N = 8$, and drew each channel matrix $\mathbf{H}_n$ according to an i.i.d. Gaussian distribution. Our plots are generated from 10000 realizations.

In Figure 2, we consider the performance of both, AM and GM, receivers with $M_{\text{Tx}} = 2$ transmitters and $M_{\text{Rx}} = 2$ receive antennas in the case of 0.1 outage probability. We plotted the performance of capacity-achieving joint ML decoding, IF decoding, and MMSE decoding. Note that IF decoding is nearly the same as MMSE decoding. Finally, we also included curves for successive integer-forcing (SIF) [3]. Recall that this refers to the technique of using decoded linear combinations to reduce the effective noise encountered in subsequent decoding steps. The static case first appeared in [3] and the AM SIF case follows similarly. Note that AM and GM SIF decoding outperforms both AM and GM MMSE significantly.

In Figure 3, we plotted the CDF of the outage rate at a fixed SNR point (SNR = 25dB). Note that, for the AM case, the difference in performance between the IF and MMSE receivers is more pronounced in the moderate outage probability regime. In Figure 4, we compare both, the IF and SIF, receivers in both fading cases. The remarkable difference between AM IF and AM SIF receivers, relative to the static case, emphasizes the importance of successive cancellation for IF receivers in the block fading case.

Finally, Figure 5 shows the performance of AM linear and IF receivers for $M_{\text{Tx}} = 3$ transmitters and $M_{\text{Rx}} = 3$ receivers in the case of 0.1 outage probability.

VI. Conclusion and Future Work

We proposed two receivers for integer-forcing for the block fading channel model, which outperform conventional linear receivers. One open question is whether one can connect the achievable rates of the GM IF receiver to a fundamental lattice quantity, such as the successive minima, in order to develop analytical bounds as in [3]. Another interesting problem is to

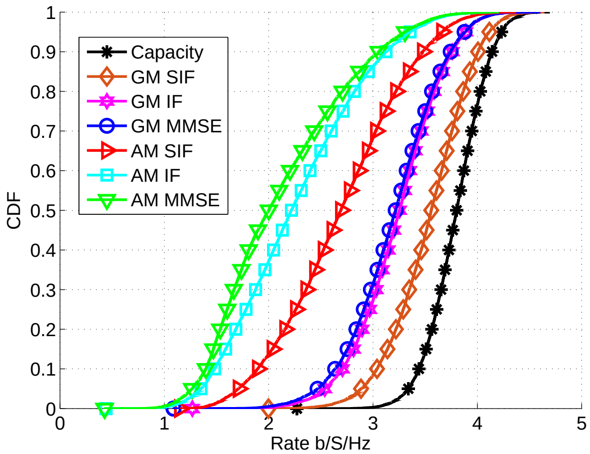

Fig. 3. The CDF of linear and IF receivers at $\mathsf{SNR} = 25$ dB over $N = 8$ subblocks with $M_{\text{Tx}} = 2$ transmitters and $M_{\text{Rx}} = 2$ receive antennas.

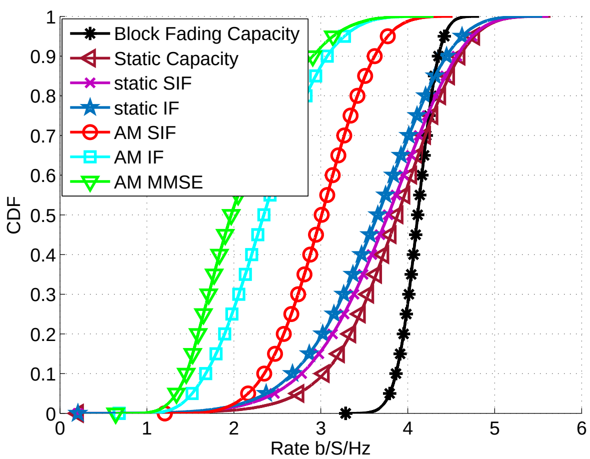

Fig. 4. The CDF of linear and IF receivers at $\mathsf{SNR} = 25$ dB over $N = 8$ subblocks with $M_{\text{Tx}} = 3$ transmitters and $M_{\text{Rx}} = 3$ receive antennas.

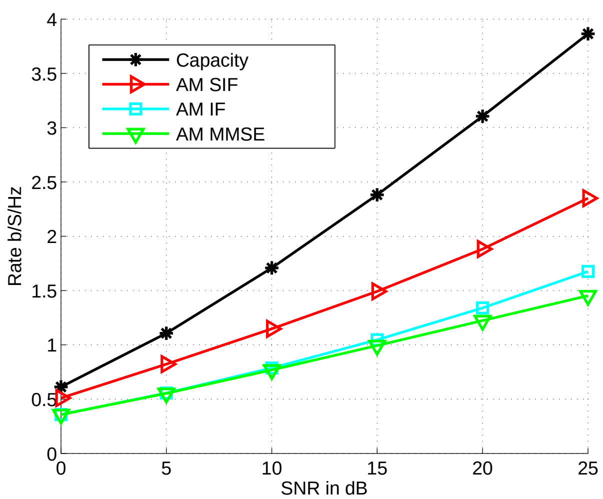

Fig. 5. Comparison of linear and IF receivers at outage probability $\rho = 0.1$ over $N = 8$ subblocks with $M_{\text{Tx}} = 3$ transmitters and $M_{\text{Rx}} = 3$ receive antennas.

develop a lattice reduction algorithm for finding the optimal integers for the GM IF receivers.

Appendix: Achievability Proof for GM IF

Due to space limitations, we only provide a sketch for the proof. Specifically, consider the following effective channel model: $\mathbf{y} = \boldsymbol{\lambda} + \mathbf{z}$ where $\boldsymbol{\lambda}$ is a lattice codeword of length T, $\mathbf{z} = [\mathbf{z}_1^{\mathsf{T}} \ \cdots \ \mathbf{z}_N^{\mathsf{T}}]^{\mathsf{T}}$, and each $\mathbf{z}_n$ is a length-(T/N) i.i.d. Gaussian vector with mean 0 and variance σ_n^2. The main difficulty is that the noise variances $\sigma_1^2, \dots, \sigma_N^2$ are not known a priori. Specifically, each noise variance is arbitrarily chosen from the interval $[0, \sigma_{\text{max}}^2]$. (In our problem setting, we can choose σ_{max}^2 such that the probability that the effective noise variances fall outside the interval has a negligible impact on the outage probability.)

As a first step, we quantize the interval $[0, \sigma_{\text{max}}^2]$ to precision δ. Let $\hat{\sigma}_1^2, \dots, \hat{\sigma}_N^2$ denote noise variances after quantization. For each of the $(\frac{\sigma_{\text{max}}^2}{\delta})^N$ possible quantized noise variances, we select an ambiguity decoder following the methodology in [11], [12]. For any $0 < \sigma_{\text{GM}}^2 < \mathsf{SNR}$, it can be shown that the average probability of error across the standard Construction A ensemble vanishes exponentially fast so long as $\left(\prod_{n=1}^N \sigma_n^2\right)^{1/N} + N\delta < \sigma_{\text{GM}}^2$. We can now apply the union bound to show that the average probability of error vanishes exponentially fast so long as $\left(\prod_{n=1}^N \sigma_n^2\right)^{1/N} + N\delta < \sigma_{\text{GM}}^2$.

Finally, it can be argued that there exists one good sequence of nested lattice codebooks with rate approaching

$$\frac{1}{2} \log \left(\frac{\mathsf{SNR}}{\sigma_{\text{GM}}^2} \right) \tag{17}$$

so long as $\left(\prod_{n=1}^N \sigma_n^2\right)^{1/N} < \sigma_{\text{GM}}^2$.

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