Abstract—It is by now well-known that significant gains are possible in network communication if users exploit the multi-access and broadcast aspects of the wireless medium through cooperation. Many cooperative schemes focus on how to exploit the statistical dependencies between transmitted and received signals. However, it has recently become clear that it is sometimes advantageous to exploit the algebraic structure of the signals through the use of structured codes. Here, we develop a rateless scheme for relays in a network to gather equations of transmitted messages to pass along to the destinations. Equations whose coefficients are closer to the fading coefficients can be decoded sooner by the relays.

I. INTRODUCTION

Consider a scenario wherein several wireless transmitters communicate across an unknown channel with the help of some relays. One possible approach is to transmit using fixed rate codes and have the relays attempt to decode messages and pass them towards the destination(s). However, when extracting a single message, a relay inherently treats other messages as noise so this strategy is interference-limited. In previous work, we have shown that relays can exploit the interference by instead decoding linear equations of messages based on channel state information at the receiver [1].

Since the fading coefficients are unknown at the transmitters, both of these strategies (decode-and-forward and compute-and-forward) include some probability of outage. Such outages can be avoided through a rateless coding framework which allows each relay to wait until it has accumulated enough information to decode. Several studies have considered rateless codes for static channels from an information theoretic perspective, including [2]–[5].

In this note, we develop a rateless framework for compute-and-forward over static channels. Relays listen until they can accumulate linear combinations (rather than mutual information) until they can recover all the messages.

II. PROBLEM STATEMENT

Our main goal is to make linear combinations available to the relays at the highest possible rates. These can in turn be sent to the destination to recover the original messages [1]. For ease of presentation, we only consider real-valued channels. However, using the techniques in [1], these results can be easily extended to complex-valued channels as well.

Let $\mathbb{R}$ denote the real field and $\mathbb{F}_p$ denote the finite field of size $p$ where $p$ is always assumed to be prime. Let $+$ denote addition over the real field and $\oplus$ addition over the finite field. Furthermore, let $\sum$ denote summation over the real field and $\bigoplus$ denote summation over the finite field. We assume that the log operation is with respect to base 2.

Definition 1: Each transmitter (indexed by $\ell = 1, 2, \ldots, L$) has a message which is a length-$k$ vector over a prime-size finite field, $w_\ell \in \mathbb{F}_p^k$. We each $w_\ell$ is generated i.i.d. according to a uniform distribution.

Definition 2: Each transmitter is equipped with an encoder, $E_\ell$, that maps its messages to a length $n_{\text{MAX}}$ codeword:

$$E_\ell : \mathbb{F}_p^k \rightarrow \mathbb{R}^{n_{\text{MAX}}}$$

for $\ell = 1, 2, \ldots, L$.

Definition 3: Each transmitter’s length-$n_{\text{MAX}}$ channel input, $x_\ell = E_\ell(w_\ell)$, is subject to the usual power constraint:

$$\frac{1}{n_{\text{MAX}}} \|x_\ell\|^2 \leq \text{SNR}$$

for $\text{SNR} \geq 0$ and $\ell = 1, 2, \ldots, L$. 

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Fig. 1. $L$ transmitters reliably communicate linear functions of the form $u_m = \bigoplus_{\ell=1}^{L} a_{m\ell} w_\ell$ to $M$ relays.
Definition 4: Each relay observes a noisy linear combination of the transmitted signals through the channel:
\[ y_m = \sum_{\ell=1}^{L} h_{m\ell}x_{\ell} + z_{m} \]  
for \( m = 1, 2, \ldots, M \) where \( h_{m\ell} \in \mathbb{R} \) are the channel coefficients and \( z \) is i.i.d. Gaussian noise, \( z \sim \mathcal{N}(0, \sigma^{2}_{mM} \cdot \mathbf{m}^{n_{\text{MAX}}}) \). Let \( h_{m} = [h_{m1} \cdots h_{mL}]^{T} \) denote the vector of channel coefficients to relay \( m \). We assume the transmitters have no channel state information and that relay \( m \) knows its own channel vector \( h_{m} \).

Definition 5: The goal of each relay is to reliably recover a linear combination of the transmitted messages. Relay \( m \) selects coefficients \( q_{m\ell} \in \mathbb{F}_{p} \) for \( \ell = 1, 2, \ldots, L \) and attempts to decode an equation:
\[ u_{m} = \bigoplus_{\ell=1}^{L} q_{m\ell}w_{\ell}. \]

Definition 6: Each relay is equipped with a variable-length decoder that can produce an estimate of one (or more) linear combination(s) of the messages at any time \( n_{\text{MIN}} \leq n \leq n_{\text{MAX}} \):
\[ \mathcal{D}(n) : \mathbb{R}^{n} \to \mathbb{F}_{p}^{k}. \]

Although our desired equations are evaluated over the finite field \( \mathbb{F}_{p} \), the channel operates over the real field \( \mathbb{R} \). Our coding scheme will allow us to efficiently exploit the channel for reliable computation if the desired equation coefficients are close to the channel coefficients in an appropriate sense. The definition below provides an embedding from the finite field to the reals that will be useful in quantifying this closeness. First, let \( g : \mathbb{F}_{p} \to \{0, 1, 2, \ldots, p-1\} \) be an embedding function that maps each element in \( \mathbb{F}_{p} \) to the corresponding element in \( \mathbb{Z}_{p} \). If \( g \) or its inverse \( g^{-1} \) are applied to a vector we assume they operate element-wise.

Definition 7: The equation with coefficient vector \( a_{m} = [a_{m1} a_{m2} \cdots a_{mL}]^{T} \in \mathbb{Z}^{L} \) is the linear combination of the transmitted messages with coefficients given by
\[ q_{m\ell} = g^{-1}([a_{m\ell}] \mod p). \]

Definition 8: We say that the equation with coefficient vector \( a_{m} \in \mathbb{Z}^{L} \) is decoded with average probability of error \( \epsilon \) if:
\[ \hat{u}_{m} \triangleq \mathcal{D}(n)(y_{m}) \]
\[ P(\hat{u}_{m} \neq u_{m}) < \epsilon. \]

Definition 9: We say that the computation rate profile \( R(h_{m}, a_{m}) \) is achievable if for any \( \epsilon > 0 \) and \( k, n_{\text{MIN}}, \) and \( n_{\text{MAX}} \) large enough, there exist encoders, \( \mathcal{E}_{1}, \ldots, \mathcal{E}_{L} \), and a decoder \( D(n) \) such that for any channel vector \( h_{m} \in \mathbb{R}^{L} \) and coefficient vectors \( a_{m} \in \mathbb{Z}^{L} \), a relay can recover its desired equation with average probability of error \( \epsilon \) if:
\[ \frac{k}{n} \log_{2} p < R(h_{m}, a_{m}) - \epsilon. \]

The above definition means that each relay is free to decode whenever it chooses. The longer it waits, the more choices of equations will become available to it.

III. NESTED LATTICE CODES

We now provide some necessary definitions from [6] on nested lattice codes. Note that all of these definitions are given over \( \mathbb{R}^{n} \).

A. Lattice Definitions

Definition 10: An \( n \)-dimensional lattice, \( \Lambda \), is a set of points in \( \mathbb{R}^{n} \) such that if \( x, y \in \Lambda \), then \( x + y \in \Lambda \), and if \( x \in \Lambda \), then \( -x \in \Lambda \). A lattice can always be written in terms of a lattice generator matrix \( L \in \mathbb{R}^{n \times n} \):
\[ \Lambda = \{x = Lw : w \in \mathbb{Z}^{n}\}. \]

Note that the origin is always a point in the lattice. A lattice \( \Lambda \) is said to be nested in a lattice \( \Lambda_{1} \) if \( \Lambda \subseteq \Lambda_{1} \).

Definition 11: A lattice quantizer is a map, \( Q_{\Lambda} : \mathbb{R}^{n} \to \Lambda \), that sends a point, \( x \), to the nearest lattice point in Euclidean distance:
\[ Q_{\Lambda}(x) = \arg \min_{\lambda \in \Lambda} ||x - \lambda||. \]

Definition 12: The fundamental Voronoi region, \( \mathcal{V} \), of a lattice, is the set of all points in \( \mathbb{R}^{n} \) that are closest to the zero vector: \( \mathcal{V} = \{x : Q_{\Lambda}(x) = 0\} \). Let \( \text{Vol}(\mathcal{V}) \) denote the volume of \( \mathcal{V} \).

Definition 13: Let \( [x] \mod \Lambda \) denote the quantization error of \( x \in \mathbb{R}^{n} \) with respect to the lattice \( \Lambda \):
\[ [x] \mod \Lambda = x - Q_{\Lambda}(x). \]

Definition 14: A nested lattice code \( \mathcal{L} \) is the set of all points of a fine lattice \( \Lambda_{1} \) that are within the fundamental Voronoi region \( \mathcal{V} \) of a coarse lattice \( \Lambda \):
\[ \mathcal{L} = \Lambda_{1} \cap \mathcal{V} = \{x : x = \lambda \mod \Lambda, \lambda \in \Lambda_{1}\}. \]

The rate of a nested lattice code is:
\[ R = \frac{1}{n} \log |\mathcal{L}| = \frac{1}{n} \log \frac{\text{Vol}(\mathcal{V})}{\text{Vol}(\mathcal{V}_{1})}. \]

Our scheme relies on mapping messages from a finite field to codewords from a nested lattice code. The relay will first decode an integer combination of lattice codewords and then convert this into an equation of the messages.

Definition 15: A lattice equation \( v \in \mathcal{L} \) is an integer combination of lattice codewords \( t_{\ell} \in \mathcal{L} \) modulo the coarse lattice:
\[ v = \left[ \sum_{\ell=1}^{L} a_{\ell}t_{\ell} \right] \mod \Lambda \]
for some coefficients \( a_{\ell} \in \mathbb{Z} \).
**Definition 16:** The second moment of a lattice $\Lambda$ is defined as the second moment per dimension of a uniform distribution over the fundamental Voronoi region $\mathcal{V}$:

$$
\sigma_\Lambda^2 = \frac{1}{n\text{Vol}(\mathcal{V})} \int_\mathcal{V} \|x\|^2 dx.
$$

(16)

The normalized second moment of a lattice is given by:

$$
G(\Lambda) = \frac{\sigma_\Lambda^2}{(\text{Vol}(\mathcal{V}))^{2/n}}
$$

(17)

**Definition 17:** A sequence of lattices $\Lambda^{(n)} \subset \mathbb{R}^n$ is good for mean-squared error (MSE) quantization if:

$$
\lim_{n \to \infty} G(\Lambda^{(n)}) = \frac{1}{2\pi e}.
$$

(18)

**Definition 18:** Let $\mathbf{z}$ be a length-$n$ random vector with distribution $\mathcal{N}(0, \sigma_Z^2 I_{n \times n})$. The volume-to-noise ratio of a lattice is given by:

$$
\mu(\Lambda, P_e) = \frac{(\text{Vol}(\mathcal{V}))^{2/n}}{\sigma_Z^2}
$$

(19)

where $\sigma_Z^2$ is chosen such that $\Pr\{\mathbf{z} \notin \mathcal{V}\} = P_e$. A sequence of lattices $\Lambda^{(n)} \subset \mathbb{R}^n$ is AWGN good if

$$
\lim_{n \to \infty} \mu(\Lambda^{(n)}, P_e) = 2\pi e \quad \forall P_e \in (0,1)
$$

(20)

and for fixed volume-to-noise ratio greater than $2\pi e$, the probability of decoding errors decays exponentially in $n$.

**Lemma 1 (Erez-Litsyn-Zamir):** There exists a sequence of lattices $\Lambda^{(n)}$ that is simultaneously quantization and AWGN good. In fact, their lattices are also good for packing and covering. See [7] for more details.

**B. Lattice Constructions**

In order to approach the computation rate profile, we need to construct a nested lattice code that is quantization and AWGN good at multiple decoding operations. Our approach is based on a suggestion from a paper by Zamir, Shamai, and Erez for lowering the implementation complexity of a nested lattice code [8]. They suggest building up the coarse lattice $\Lambda \subset \mathbb{R}^{n_{\text{MAX}}}$ using a Cartesian product of a smaller lattice $\Lambda' \subset \mathbb{R}^\tau$ that has the desired properties:

$$
\Lambda = \Lambda' \times \cdots \times \Lambda'
$$

(21)

where we assume that $n_{\text{MAX}} = \tau B$ for some $B \in \mathbb{Z}_+$. We will then find a fine lattice $\Lambda_1$ such that $\Lambda \subset \Lambda_1$. The relay can listen into these “chunks” of length $\tau$ and decode when it has accumulated enough chunks to satisfy the rate requirement of the desired linear equation. The choice of $\Lambda$ allows us to show that the fine lattice codewords at length $b\tau$ are just the lattice codewords from length $(b-1)\tau$ concatenated with a new chunk.

We will select the smaller lattice $\Lambda'$ using Lemma 1. The following two lemmas relate the fundamental Voronoi regions, normalized second moments, and volume-to-noise ratios of $\Lambda$ and $\Lambda'$. For notational convenience, we define

$$
x^{[b]} = \left[x[(b-1)\tau + 1], x[(b-1)\tau + 2], \ldots, x[b\tau]\right]^T
$$

(22)

$$
x^{[1:b]} = \left[x[1], x[2], \ldots, x[b\tau]\right]^T.
$$

(23)

**Lemma 2:** Let $\mathcal{V}'$ and $\mathcal{V}$ be the fundamental Voronoi regions of $\Lambda'$ and $\Lambda$, respectively. We can write $\mathcal{V}$ as the Cartesian product of $\mathcal{V}'$:

$$
\mathcal{V} = \mathcal{V}' \times \cdots \times \mathcal{V}'
$$

(24)

which implies that $\text{Vol}(\mathcal{V}) = (\text{Vol}(\mathcal{V}'))^B$.

**Lemma 3:** Let $G(\Lambda')$ and $G(\Lambda)$ be the normalized second moments of $\Lambda'$ and $\Lambda$, respectively. Then, we have that $G(\Lambda) = G(\Lambda')$.

**Proof:** We have that:

$$
\int_{\mathcal{V}} \|x\|^2 dx = \sum_{b=1}^B \int_{\mathcal{V}} \|x^{[b]}\|^2 dx
$$

(25)

$$
= \sum_{b=1}^B \int_{\mathcal{V}'} \cdots \int_{\mathcal{V}'} dx^{[1:b-1]} dx^{[b+1:B]} \int_{\mathcal{V}'} \|x^{[b]}\|^2 dx^{[b]}
$$

(26)

$$
= \sum_{b=1}^B (\text{Vol}(\mathcal{V}'))^{B-1} \int_{\mathcal{V}'} \|x^{[b]}\|^2 dx^{[b]}
$$

(27)

$$
= B (\text{Vol}(\mathcal{V}'))^{B-1} \int_{\mathcal{V}'} \|x^{[1]}\|^2 dx^{[1]}
$$

(28)

Using this, we can relate the second moment of $\Lambda$ to the second moment of $\Lambda'$:

$$
\sigma_{\Lambda}^2 = \frac{B (\text{Vol}(\mathcal{V}'))^{B-1}}{n_{\text{MAX}} \text{Vol}(\mathcal{V})} \int_{\mathcal{V}'} \|x^{[1]}\|^2 dx^{[1]}
$$

(29)

$$
\frac{1}{\tau \text{Vol}(\mathcal{V}')} \int_{\mathcal{V}'} \|x^{[1]}\|^2 dx^{[1]} = \sigma_{\Lambda'}^2.
$$

(30)

since $n_{\text{MAX}} = \tau B$ and $\text{Vol}(\mathcal{V}) = (\text{Vol}(\mathcal{V}'))^B$. Finally, we get that:

$$
G(\Lambda) = \frac{\sigma_{\Lambda}^2}{(\text{Vol}(\mathcal{V}))^{2/n_{\text{MAX}}}} = \frac{\sigma_{\Lambda'}^2}{(\text{Vol}(\mathcal{V}'))^{2/\tau}} = G(\Lambda')
$$

(31)

**Lemma 4:** Let $\mu(\Lambda', P_e)$ and $\mu(\Lambda, P_e)$ be the volume-to-noise ratios of $\Lambda'$ and $\Lambda$, respectively. The following upper bound holds:

$$
\mu(\Lambda, P_e) \leq \mu \left(\Lambda', \frac{P_e}{B}\right)
$$

(32)

**Proof:** Let $\mathbf{z} \sim \mathcal{N}(0, \sigma_Z^2 I_{n_{\text{MAX}} \times n_{\text{MAX}}})$ where $\sigma_Z^2$ is chosen such that $\Pr(\mathbf{z}^{[b]} \notin \mathcal{V}') = P_e' > 0$. By the union bound,

$$
\Pr(\mathbf{z} \notin \mathcal{V}) < \sum_{b=1}^B \Pr(\mathbf{z}^{[b]} \notin \mathcal{V}').
$$

(33)

This implies that

$$
\mu(\Lambda, BP_e') \leq \frac{(\text{Vol}(\mathcal{V}'))^{2/\tau}}{\sigma_Z^2} = \mu(\Lambda', P_e').
$$
again using \( \eta_{\text{MAX}} = \tau B \) and \( \text{Vol}(Y) = \text{(Vol}(Y'))^{B} \).

By construction, the lattice generator matrix \( L \) of \( \Lambda \) is block diagonal with the lattice generator matrix \( L' \) of \( \Lambda' \) repeated on the diagonal, \( L = \text{diag}(L' L' \cdots L') \).

Scale \( \Lambda \) such that its second moment is equal to SNR. Our fine lattice \( \Lambda_1 \) is defined as follows:

1. Draw a matrix \( G \in \mathbb{R}^{B \times k} \) with every element chosen i.i.d. according to the uniform distribution over \( \mathbb{F}_p \).
2. Define the codebook \( C = \{ c = Gw : w \in \mathbb{F}_p^k \} \).
3. Form the lattice \( \Lambda_1 \) by projecting the codebook into the reals by \( g(\cdot) \), scaling down by a factor of \( p \), and placing a copy at every integer vector. This tiles the codebook over \( \mathbb{R}^n \), \( \Lambda_1 = p^{-1}G(C) + \mathbb{Z}^n \).
4. Rotate \( \Lambda_1 \) by the lattice generator matrix \( L \) of \( \Lambda \) to get the fine lattice \( \Lambda = L\Lambda_1 \).

Let \( G^{[b]} \) denote the submatrix of \( G \) that includes all columns and rows from \( 1 \) to \( (b-1)\tau \) to \( b\tau \). Also, let \( G^{[1:b]} \) denote the submatrix of \( G \) that includes all columns and all rows from \( 1 \) to \( b\tau \) and let \( \Lambda^{[1:b]} \) and \( \mathcal{V}^{[1:b]} \) denote the first \( b\tau \) coordinates of \( \Lambda \) and \( \mathcal{V} \), respectively. Since \( L \) is block diagonal, for any fine lattice point \( t \in \Lambda_1 \), each chunk \( t^{[b]} \) can be viewed as generated from \( G^{[b]} \), embedded in the reals, and rotated by \( L' \). Note that this construction guarantees that \( \Lambda^{[1:b]} \subset \Lambda_1^{[1:b]} \).

We need that \( G^{[1:b]} \) is full rank for all \( b = 1, \ldots, B \). By the union bound, the probability that \( G^{[1:b]} \) is not full rank for some \( b \) is upper bounded by:

\[
\sum_{b=1}^{B} \sum_{w \in \mathbb{F}_p^k \setminus \{0\}} \Pr \{ G^{[1:b]}w = 0 \} \leq (p^k - 1) \sum_{b=1}^{B} p^{-B\tau} = (p^k - 1) \frac{p^{-\tau}(1 - p^{-B\tau})}{1 - p^{-\tau} - 1} \quad (34)
\]

Since \( k \) will be chosen to grow slower than \( \tau \), all submatrices \( G^{[1:b]} \) are full rank with probability that goes to \( 1 \) as \( \tau \to \infty \). Note that if \( G^{[1:b]} \) has full rank, then the number of fine lattice points in the fundamental Voronoi region \( \mathcal{V}^{[1:b]} \) of the coarse lattice is given by \( |\Lambda^{[1:b]} \cap \mathcal{V}^{[1:b]}| = p^k \) and the rate after chunk \( b \) is \( R_b = \frac{\log p}{\tau} \). Let \( \Lambda_1^{(i)} \) denote the \( i \)th point in \( \Lambda_1^{[1:b]} \cap \mathcal{V}^{[1:b]} \) for \( i = 0, 1, 2, \ldots, p^k - 1 \). From [7], we have that:

- \( \Lambda_1^{[1:b]}(i) \) is uniformly distributed over \( p^{-1}\Lambda^{[1:b]} \cap \mathcal{V}^{[1:b]} \).
- For any \( i_1 \neq i_2, [\Lambda_1^{[1:b]}(i_1) - \Lambda_1^{[1:b]}(i_2)] \) mod \( \Lambda \) is uniformly distributed over \( p^{-1}\Lambda^{[1:b]} \cap \mathcal{V}^{[1:b]} \).

IV. Achievable Scheme

First, we assume that the relay cannot decode until it has received at least \( b_{\text{START}} \) chunks so that \( \eta_{\text{MIN}} = \tau b_{\text{START}} \). We pick a maximum rate \( R_{\text{MAX}} = \frac{k \log p}{\tau b_{\text{START}}} \) and choose \( b_{\text{START}} \) large enough so that \( \frac{k \log p}{\tau b_{\text{START}}} = R_{\text{MAX}} \leq \frac{R_{\text{MAX}} \cdot \text{SNR}}{B} \). This ensures that the rate from one chunk to the next does not decrease by more than \( \frac{1}{2} \). From [6], we need \( p \) to be a prime larger than \( \tau \log \tau \) to guarantee the fine lattice will be AWGN good. Setting \( k = \left\lfloor \frac{R_{\text{MAX}} \cdot \text{SNR}}{B} \right\rfloor \) ensures that the rate after \( b_{\text{START}} \) chunks is at most \( R_{\text{MAX}} \) (and at least \( R_{\text{MAX}} + \frac{1}{2} \)). It can be shown that \( R_{\text{MIN}} = \frac{k \log p}{\tau b_{\text{START}}} \leq \frac{R_{\text{MAX}} \cdot \text{SNR}}{B} \). Finally, for some \( \delta > 0 \) we select \( \tau \) large enough so that \( G'(\Lambda') = \frac{1}{2\pi} \) and \( \mu' = (1 + \delta)2\pi \epsilon \). Using Lemmas 3 and 4, we get that \( G'(\Lambda') = \frac{1}{2\pi} \) and \( \mu'(\Lambda') = (1 + \delta)2\pi \epsilon \).

We now develop an achievable computation rate profile using nested lattice codes. First, each encoder maps its finite-field message vector onto a lattice point \( \Lambda_1 \cap \mathcal{V} \). These lattice points are transmitted across the channel. When the decoder has observed enough channel outputs, it decodes an equation of lattice points and maps this back to an equation over the finite field. The following lemma gives maps between the finite field and the nested lattice code.

**Lemma 5:** Let \( \mathbf{w}_t \in \mathbb{F}_p^k \) be the message at encoder \( t \). There exists a bijective function \( \phi : \mathbb{F}_p^k \to \Lambda_1 \cap \mathcal{V} \) such that if \( \mathbf{w}_t = \phi(\mathbf{w}_t) \) then for any \( \mathbf{a}_m \in \mathbb{Z}^n \),

\[
\phi^{-1}(\sum_{t=1}^{L} a_m t_d) = \bigoplus_{\ell=1}^{L} q_m \mathbf{w}_\ell \quad (35)
\]

\[
q_m = g^{-1}([a_m] \mod p) \cdot \epsilon \quad (36)
\]

See Lemmas 5 and 6 in [1] for a proof.

**Theorem 1:** The following computation rate profile is achievable:

\[
R(h_m, a_m) = \max \left\{ 0, \min \left( f(h_m, a_m), R_{\text{MAX}} \right) \right\} \quad (37)
\]

\[
f(h_m, a_m) = \frac{1}{2} \log \left( \frac{\|a_m\|^2 - \text{SNR} \|h_m a_m\|^2}{1 + \text{SNR} \|h_m a_m\|^2} \right)^{-1} \quad (38)
\]

**Proof:** Each encoder maps its message \( \mathbf{w}_t \) into a lattice point \( \mathbf{t}_t = \phi(\mathbf{w}_t) \) using Lemma 5. Also, each encoder is given a dither vector \( d \), which is independently drawn according to a uniform distribution over \( \mathcal{V} \). All dither vectors are made available to each relay. The channel input from encoder \( \ell \) is \( x_t = [\mathbf{t}_t - d] \) mod \( \Lambda \). The \( x_t \) are independent and uniform over \( \mathcal{V} \) so \( \frac{\text{max}}{\text{max}} E[\|x_t\|^2] = \text{SNR} \), with expectation taken over the dithers. Let \( Q_1^{[1:b]} \) denote the lattice quantizer for \( \Lambda_1^{[1:b]} \). Relay \( m \) observes \( y_m \) up to time \( n \) (greater than \( \tau_{\text{START}} \)) and wants to decode the equation with coefficient vector \( a_m \). Set \( b = \lceil n/\tau \rceil \) and \( \alpha_m = \frac{\text{SNR} h_m^T a_m}{1 + \text{SNR} \|h_m a_m\|^2} \). From \( y_m^{[1:b]} \) the relay computes:

\[
s_m = \alpha_m y_m + \sum_{\ell=1}^{L} a_m d_\ell^{[1:b]} \quad (39)
\]

To get an estimate of the lattice equation \( v_m = \sum_{\ell} a_m w_\ell \), we quantize onto \( \Lambda_1^{[1:b]} \) modulo the coarse lattice \( \Lambda_1^{[1:b]} \):

\[
\hat{v}_m = \left[ Q_1^{[1:b]}(v_m + z_{eq,m}) \right] \mod \Lambda_1^{[1:b]} \quad (40)
\]
where the density of $z_{eq,m}$ approaches the density of a length $b\tau$ i.i.d. zero-mean Gaussian vector with variance

$$N_{eq,m} = (\alpha_m)^2 + \text{SNR}||\alpha_m h_m - a_m||^2$$

as $\tau \to \infty$ (see [1] for more details).

If instead of the fine lattice code we used a random code with elements chosen according to a uniform distribution over $p^{-1}\Lambda[1:b] \cap \mathcal{Y}[1:b]$, we could show that this code is AWGN good using Lemma 4. Since the fine lattice points are chosen uniformly and pairwise independently over $p^{-1}\Lambda[1:b] \cap \mathcal{Y}[1:b]$, then, by the union bound, we can show $\Lambda[1:b]$ is AWGN good for all $b$ with probability at least $1 - \frac{1}{\delta}$ for $\tau$ large enough.

If $G[1:b]$ is full rank, the rate after $b$ chunks is $R[b] = \frac{b}{b\tau} \log p$. We choose $\tau$ large enough such that the probability that $G[1:b]$ is full rank for all $b$ is greater than $1 - \frac{1}{\delta}$. We now condition on $G[1:b]$ being full rank and use (14) to solve for the volume of $\mathcal{Y}[1:b]$ in terms of $R[b]$ and the volume of $\mathcal{Y}[1:b]$: 

$$\text{Vol}(\mathcal{Y}[1:b]) = \text{Vol}(\mathcal{Y}[1:b])^{2-\log p} = (\text{Vol}(\mathcal{Y}[1:b]))^{b-2\log p}.$$ 

From (17), we get that:

$$\text{Vol}(\mathcal{Y}[1:b]) = \left(\frac{\sigma^2}{C(G[1:b])}\right)^{\tau/2} = \left(\frac{2\pi e \text{SNR}}{1 + \delta}\right)^{\tau/2}$$

and

$$\text{Vol}(\mathcal{Y}[1:b]) = \left(\frac{2\pi e \text{SNR}}{1 + \delta}\right)^{b\tau/2} = 2^{k\log p}.$$ 

We make an error, $v_m \neq v_m$, if $z_{eq,m}$ leaves the Voronoi region so the probability of error is just $P_e = \text{Pr}(z_{eq,m} \notin \mathcal{V}[1:b]) = \frac{1}{3\tau}$. Recall that by Lemma 5, $\phi^{-1}(v_m) = u_m$. Choose some $\gamma > 0$. Assuming $\Lambda[1:b]$ is AWGN good, we have that $\mu(\Lambda[1:b], P_e) = (1 + \gamma)2\pi e$ for $\tau$ large enough. Using (19), we have:

$$\mu(\Lambda[1:b], P_e) = \left(\text{Vol}(\mathcal{Y}[1:b])\right)^{2/b\tau} = \left(\frac{2\pi e \text{SNR}}{1 + \delta}\right)^{2/b\tau}$$

Using the compute-and-forward strategy in this paper, we determine the expected time until the relay has acquired equations of a certain rank. For comparison, we also find the expected time of a rudimentary decode-and-forward strategy that recovers messages individually while treating others as noise (the number of messages corresponds to the rank). For simplicity, we disallow successive interference cancellation for both strategies. In Figure 2, we plot the ratio of the expected times and note that compute-and-forward is significantly faster when the relay does not need a full rank set of equations.

Fig. 2. Ratio of expected times for decode-and-forward and compute-and-forward until a relay can recover equations with a given rank.

We now provide a simple comparison between compute-and-forward and decode-and-forward. There is a single relay that observes $5$ users through a channel with fading coefficients generated i.i.d. according to a normal distribution $h_f \sim \mathcal{N}(0, 1)$. The goal is for the relay to decode equations of the original messages with the highest possible rank. As time progresses, the relay will be able to get more linearly independent equations. This is reminiscent of the network coding feedback strategy developed in [9] which has relays acknowledge recovered degrees-of-freedom instead of recovered messages.

**REFERENCES**


