Computing over Multiple-Access Channels with Connections to Wireless Network Coding

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Outline

1. Source-Channel Separation Theorems
2. Computation Coding
3. Three Illustrative Examples
4. Multicast Capacity for Finite Field MAC Networks
Source-Channel Separation Theorems (or lack thereof)

- Shannon (1948): separation optimal for point-to-point links
  - Reliable communication possible if $H(S) < \max_{p(x)} I(X; Y)$
- Cover-El Gamal-Salehi (1980): no separation theorem for multiple-access channels (MACs) with correlated sources.
- Is there a separation theorem for sending functions over MACs?
Problem Statement: Reliable Computation over MACs

- $M$ users each observe a source
- Only want a function of the sources, $U = f(S_1, S_2, \ldots, S_M)$
- Reliable computation: $\lim_{k \to \infty} P(\hat{U}^k \neq U^k) = 0$
- Computation rate, $\kappa = \frac{k}{n}$, functions sent per channel use
- No separation theorem *even if the sources are independent*
Separation-Based Scheme

- $\mathcal{R}_U$: distributed compression rate region, only known in special cases (Slepian-Wolf, Körner-Marton)
- $\mathcal{R}_{MAC}$: MAC capacity region
  - 2-user MAC: convex closure of all $(R_1, R_2)$ satisfying:
    \[
    R_1 < I(X_1; Y | X_2) \quad R_2 < I(X_2; Y | X_1) \quad R_1 + R_2 < I(X_1, X_2; Y)
    \]
- Separation-based computation possible if $\mathcal{R}_U \cap \mathcal{R}_{MAC} \neq \emptyset$
Routing Is Suboptimal

- Butterfly example
- One source, two sinks, and noiseless rate 1 links
- Max-flow min-cut bound = 2 bits per network use
- **Routing suboptimal**, only transmits 1.5 bits
- **Mixing optimal**, send $a \oplus b$ down center path
Network Coding

- **Ahlswede-Cai-Li-Yeung** (2000): for any network of bit pipes, network coding achieves the max-flow min-cut bound
- Later works have generalized the setup and simplified the coding scheme:
  - **Li-Yeung-Cai** (2003): linear network coding
  - Ho et al., Jaggi et al., Sanders et al. (2003): distributed construction, bounds on field size
- **Channel-network separation is optimal** for multicasting over a network of point-to-point channels
A Simple Network with a MAC

- Add a MAC into the usual butterfly network
- Separation can send $a$ and $b$ iff $R_{MAC}^1 \geq 1$ and $R_{MAC}^2 \geq 1$
- What if the MAC just takes the mod-2 sum: $Y = X_1 \oplus X_2$
- Uncoded transmission optimal
- Separation-based coding suboptimal
  - Must transmit $a$ and $b$ individually
  - $R_{MAC}^1 + R_{MAC}^2 \leq 1$
- What if there is noise?
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A New Coding Technique: Computation Coding

- Many MACs compute a function of the sources then add noise (ex: Gaussian MAC)
- Uncoded transmission achieves rate gains but incurs a noise penalty
- Trick: Use structured codes
- Example: Mod-2 Noisy Adder MAC (M2MAC)

\[
\begin{align*}
S_1 \rightarrow & \quad ENC_1 \quad X_1 \quad Z \\
S_2 \rightarrow & \quad ENC_2 \quad X_2 \\
U = S_1 \oplus S_2 \\
\end{align*}
\]
Computation Coding for the M2MAC: Proof Sketch

- Only want $U = S_1 \oplus S_2$ at the decoder
- **Key idea:** Use the same linear codebook at each encoder
- Pick good source and channel coding matrices $H$ and $G$

$$
x_1 = s_1HG \\
x_2 = s_2HG \\
y = s_1HG \oplus s_2HG \oplus z = uHG \oplus z
$$

- After the channel, looks as if $U$ was jointly encoded
- Relies on low complexity codes
Example 1: Butterfly Network with an M2MAC

- Add the M2MAC into a butterfly network
- Separation-based scheme requires:

\[ H(S) < 1 + \left( \frac{1 - h_B(p)}{2} \right) \]

- Computation coding is optimal and meets the max-flow min-cut bound:

\[ H(S) < 1 + (1 - h_B(p)) \]
Example 2: Not Just MACs over Fields

- Butterfly network with real-adder MAC
- Channel output is $X_1 + X_2$ passed through a symmetric DMC
- Example: Crossover probability 0.2
- Separation-based scheme requires:
  \[ H(S) < 1.33 \]
- Computation coding scheme:
  \[ H(S) < 1.40 \]
Scheme: Transform Real-Adder to Mod-2 Adder

- Real-adder MAC is nearly a mod-2 adder
- Just map output symbol 2 to 0
- Uniform distribution is a good input distribution
What about Gaussian?

- Butterfly network with Gaussian MAC:
  \[ Y = X_1 + X_2 + Z, \quad Z \sim \mathcal{N}(0, \frac{1}{3}) \]
- Usual power constraint on the inputs, \( P = 1 \)
- Mapping to a mod-2 adder MAC does worse than separation
- Reason: bad input distribution
- Need a better strategy
Systematic Computation

User 1 \[ S_1[1] \quad S_1[2] \quad S_1[3] \quad \cdots \quad S_1[k] \quad \text{MAC coded } s_1 H \]

Channel computes noisy \( U = f(S_1, S_2) \)

Decoder refines with \( uH = s_1 H \oplus s_2 H \)

User 2 \[ S_2[1] \quad S_2[2] \quad S_2[3] \quad \cdots \quad S_2[k] \quad \text{MAC coded } s_2 H \]

- **Phase 1**: Uncoded transmission. Receiver gets a noisy version of function.
- **Phase 2**: Separation-based scheme sends linear update bins
- Binning only needs a mapping to a linear function over some field
- Trades off between using channel function and optimal MAC input distribution
Systematic Computation for the Gaussian MAC

- Butterfly network with Gaussian MAC:
  \[ Y = X_1 + X_2 + Z, \quad Z \sim \mathcal{N}(0, \frac{1}{3}) \]
- Usual power constraint on the inputs, \( P = 1 \)
- Separation-based scheme requires:
  \[ H(S) < 1.70 \]
- Systematic computation coding scheme:
  \[ H(S) < 1.76 \]
Multicasting over Finite Field MAC Networks

- Usual network coding setup: Single source, \( L \) receivers, encoder/decoder nodes, and directed point-to-point channels
- Add any number of MACs of the form:
  - \( Y = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_M X_M + Z \)
  - \( \alpha_i \in \mathbb{F} \setminus \{0\} \)
  - \( X_i, Z \in \mathbb{F} \)
  - Addition over \( \mathbb{F} \)
- All MACs operate over the same field
- No broadcast constraint

**Theorem**

*If the field size of the MACs, \(|\mathbb{F}|\), is larger than the number of receivers, \( L \), then the multicast capacity is given by the max-flow min-cut bound.*
Proof Outline

• Need result of Ho et al. (2003): Linear solution for a network of directed point-to-point links exists if the alphabet size is larger than the number of receivers

• Apply network transformation:
  • Replace each MAC with a node whose output has a capacity equal to the original sum-rate capacity
  • Each incoming link to the original MAC is replaced with an infinite capacity link to the new node

• Find a linear solution for this network

• Find a computation code for each MAC to duplicate the function of its replacement
Proof Outline
Conclusions

- New technique: computation coding
  - Sometimes optimal, often helps
  - Relies on low complexity, structured codes
- Interference useful for wireless network coding
- Structural considerations may be necessary in large networks